Chapter 15

Hashing

CS 473: Fundamental Algorithms, Spring 2011
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15.1 Hash Tables

15.2 Introduction

15.2.0.1 Dictionary Data Structure

- A universe $\mathcal{U}$ of keys that have a total order: numbers, strings, etc.
- Data structure to store a subset $S \subseteq \mathcal{U}$
- **Operations:**
  - **Search**/lookup: given $x \in \mathcal{U}$ is $x \in S$?
  - **Insert**: given $x \notin S$ add x to S.
  - **Delete**: given $x \in S$ delete x from S
- ***Static*** structure: S given in advance or changes very infrequently, main operations are lookups.
- ***Dynamic*** structure: S changes rapidly so inserts and deletes as important as lookups.

15.2.0.2 Dictionary Data Structures

Common solutions:

- Static:
  - Store S as a *sorted* array

  - Lookup: binary search in $O(\log |S|)$ time (comparisons)

- Dynamic:

  - Store S in a *balanced* binary search tree
  - Lookup, Insert, Delete in $O(\log |S|)$ time (comparisons)

15.2.0.3 Dictionary Data Structures

Question: "Should Tables be Sorted?"
(also title of famous paper by Turing award winner Andy Yao)

Hashing is a widely used & powerful technique for dictionaries.

Motivation:

- Universe $\mathcal{U}$ may not be (naturally) totally ordered
- Keys correspond to large objects (images, graphs etc) for which comparisons are very expensive
- Want to improve "average" performance of lookups to $O(1)$ even at cost of extra space or errors with small probability: many applications for fast lookups in networking, security, etc.

15.2.0.4 Hashing and Hash Tables

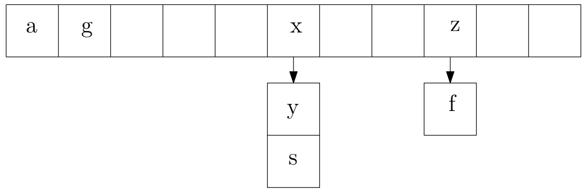
Hash Table data structure:

- A (hash) table/array T of size m (the table size)
- A hash function $h : \mathcal{U} \to \{0, \ldots, m-1\}$
- Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T

Given $S \subseteq \mathcal{U}$. How do we store S and how do we do lookups?
Ideal situation:

- Each element $x \in S$ hashes to a distinct slot in T. Store x in slot $h(x)$
- Lookup: given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

Collisions unavoidable. Several different techniques to handle them.



15.2.0.5 Handling Collisions: Chaining

Collision: $h(x) = h(y)$ for some $x \neq y$.

Chaining to handle collisions:

- For each slot i store all items hashed to slot i in a linked list. $T[i]$ points to the linked list
- Lookup: to find if $y \in \mathcal{U}$ is in T, check the linked list at $T[h(y)]$. Time proportion to size of linked list.

15.2.0.6 Handling Collisions

Several other techniques:

- Open addressing
- ...
- Cuckoo hashing

15.2.0.7 Understanding Hashing

Does hashing give $O(1)$ time per operation for dictionaries?

Questions:

- Complexity of evaluating h on a given element?
- Relative sizes of the universe $\mathcal{U}$ and the set to be stored S.
- Size of table relative to size of S.
- Worst-case vs average-case vs randomized (expected) time?
- How do we choose h?

15.2.0.8 Understanding Hashing

- Complexity of evaluating h on a given element? Should be small.
- Relative sizes of the universe $\mathcal{U}$ and the set to be stored S: typically $|\mathcal{U}| \gg |S|$.
- Size of table relative to size of S. The ***load factor*** of T is the ratio n/t where $n = |S|$ and $m = |T|$. Typically n/t is a small constant smaller than 1.

  Also known as the ***fill factor***.

Main and interrelated questions:

- Worst-case vs average-case vs randomized (expected) time?
- How do we choose h?

15.2.0.9 Single hash function

- $\mathcal{U}$: universe (very large).
- Assume $N = |\mathcal{U}| \gg m$ where m is size of table T. In particular assume $N \geq m^2$ (very conservative).
- Fix hash function $h : \mathcal{U} \to \{0, \ldots, m-1\}$.
- N items hashed to m slots. By pigeon hole principle there is some $i \in \{0, \ldots, m-1\}$ such that $N/m \geq m$ elements of $\mathcal{U}$ get hashed to i!
- Implies that there is a set $S \subseteq \mathcal{U}$ where $|S| = m$ such that all of S hashes to same slot!

Lesson: For every hash function there is a very bad set!

15.2.0.10 Picking a hash function

- Hash function are often chosen in an ad hoc fashion. Implicit assumption is that input behaves well.
- Theory and sound practice suggests that a hash function should be chosen properly with guarantees on its behavior.

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

- $\mathcal{H}$ is a *family* of hash functions: each function $h \in \mathcal{H}$ should be efficient to evaluate (that is, to compute $h(x)$)
- h is chosen *randomly* from $\mathcal{H}$ (typically uniformly at random). Implicitly assumes that $\mathcal{H}$ allows an efficient sampling.
- Randomized guarantee: should have the property that for any *fixed* set $S \subseteq \mathcal{U}$ of size m the expected number of collisions for a function chosen from $\mathcal{H}$ should be "small". Here the expectation is over the randomness in choice of h.

15.2.0.11 Picking a hash function

Question: Why not let $\mathcal{H}$ be the set of *all* functions from $\mathcal{U}$ to $\{0, 1, \ldots, m-1\}$?

- Too many functions! A random function has high complexity!

  Question: Are there good and compact families $\mathcal{H}$?

- Yes... But what it means for $\mathcal{H}$ to be good and compact.

15.3 Universal Hashing

15.3.0.12 Uniform hashing

Question: What are good properties of $\mathcal{H}$ in distributing data?

- Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \leq i < m$.

- Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between x and y should be at most $1/m$. In other words $\Pr[h(x) = h(y)] = 1/m$ (cannot be smaller).

- Second property is stronger than the first and the crucial issue.

Definition 15.3.1 *A family hash function $\mathcal{H}$ is* **2-universal** *if for all distinct $x, y \in \mathcal{U}$, $Pr[h(x) = h(y)] = 1/m$ where m is the table size.*

Note: The set of all hash functions satisfies stronger properties!

15.3.0.13 Analyzing Uniform Hashing

- T is hash table of size m.

- $S \subseteq \mathcal{U}$ is a *fixed* set of size m.

- h is chosen randomly from a uniform hash family $\mathcal{H}$.

- x is a *fixed* element of $\mathcal{U}$. Assume for simplicity that $x \notin S$.

  Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

15.3.0.14 Analyzing Uniform Hashing

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

- The time to look up x is the size of the list at $T[h(x)]$: same as the number of elements in S that collide with x under h.
- Let $\ell(x)$ be this number. We want $\mathrm{E}[\ell(x)]$
- For $y \in S$ let A_y be the even that x, y collide and D_y be the corresponding indicator variable.

15.3.1 Analyzing Uniform Hashing

15.3.1.1 Continued...

$$\ell(x) = \sum_{y \in S} D_y$$

$$\begin{aligned}
\Rightarrow \mathrm{E}[\ell(x)] &= \sum_{y \in S} \mathrm{E}[D_y] \quad \text{linearity of expectation} \\
&= \sum_{y \in S} Pr[h(x) = h(y)] \\
&= \sum_{y \in S} \frac{1}{m} \quad \text{since } \mathcal{H} \text{ is a uniform hash family} \\
&= |S|/m \leq 1
\end{aligned}$$

15.3.1.2 Analyzing Uniform Hashing

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

Answer: $O(1)$!

Comments:

- $O(1)$ expected time also holds for insertion.
- Analysis assumes static set S but holds as long as S is a set formed with at most $O(m)$ insertions and deletions.
- *Worst-case* look up time can be large! How large? $\Omega(\log n / \log \log n)$.

15.3.2 Rehashing, amortization and...

15.3.2.1 ... making the hash table dynamic

Previous analysis assumed fixed S of size $\simeq m$.

Question: What happens as items are inserted and deleted?

- If $|S|$ grows to more than cm for some constant c then hast table performance clearly degrades
- If $|S|$ stays around $\simeq m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

Solution: Rebuild hash table periodically!

- Choose a new table size based on current number of elements in table.
- Choose a new random hash function and rehash the elements.
- Discard old table and hash function.

Question: When to rebuild? How expensive?

15.3.2.2 Rebuilding the hash table

- Start with table size m where m is some estimate of $|S|$ (can be some large constant).
- If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.
- If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant c (say 10), rebuild.

Amortize cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when S changes. Hence $O(1)$ expected look up/insert/delete time *dynamic* data dictionary data structure!

15.3.2.3 Some math required...

Lemma 15.3.2 *Let p be a prime number,*
x: an integer number in $\{1, \ldots, p-1\}$.
$\implies$ *There exists a unique y s.t. $xy = 1 \mod p$.*

In other words: For every element there is a unique inverse.
$\implies$ $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ when working module p is a field.

15.3.2.4 Proof of lemma

Claim 15.3.3 *Let p be a prime number. For any $\alpha, \beta, i \in \{1, \ldots, p-1\}$ s.t. $\alpha \neq \beta$, we have that $\alpha i \neq \beta i \mod p$.*

Proof: Assume for the sake of contradiction $\alpha i = \beta i \mod p$. Then

$$\begin{aligned} & i(\alpha - \beta) = 0 \mod p \\ \implies & \quad p \text{ divides } i(\alpha - \beta) \\ \implies & \quad p \text{ divides } \alpha - \beta \\ \implies & \quad \alpha - \beta = 0 \\ \implies & \quad \alpha = \beta. \end{aligned}$$

And that is a contradiction. ∎

15.3.3 Proof of lemma...

15.3.3.1 Uniqueness.

Proof: Follows immediately from the above claim. Indeed, assume the claim is false and there are two distinct numbers $y, z \in \{1, \ldots, p-1\}$ such that

$$xy = 1 \mod p \quad \text{and} \quad xz = 1 \mod p.$$

But this contradicts the above claim (set $i = x$, $\alpha = y$ and $\beta = z$). ∎

15.3.4 Proof of lemma...

15.3.4.1 Existence

Proof: By claim, for any $\alpha \in \{1, \ldots, p-1\}$ we have that $\{\alpha * 1 \mod p, \alpha * 2 \mod p, \ldots, \alpha * (p-1) \mod$
$\{1, 2, \ldots, p-1\}$.
$\implies$ There exists a number $y \in \{1, \ldots, p-1\}$ such that $\alpha y = 1 \mod p$. ∎

15.3.4.2 Constructing Universal Hash Families

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

- Choose a ***prime*** number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ is a field.
- For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
- Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0$. Note that $|\mathcal{H}| = p(p-1)$.

Theorem 15.3.4 *$\mathcal{H}$ is a 2-universal hash family.*

Comments:

- Hash family is of small size, easy to sample from.
- Easy to store a hash function (a, b have to be stored) and evaluate it.

15.3.4.3 Constructing Universal Hash Families

Theorem 15.3.5 *$\mathcal{H}$ is a (2)-universal hash family.*

Proof: Fix $x, y \in \mathcal{U}$. What is the probability they will collide if h is picked randomly from $\mathcal{H}$?

- Let a, b be bad for x, y if $h_{a,b}(x) = h_{a,b}(y)$
- **Claim:** Number of bad pairs is at most $p(p-1)/m$.
- Total number of hash functions is $p(p-1)$ and hence probability of a collision is $\leq 1/m$.

■

15.3.4.4 Some Lemmas

Lemma 15.3.6 *If $x \neq y$ then for any $a, b \in \mathbb{Z}_p$ such that $a \neq 0$, $ax + b \mod p \neq ay + b \mod p$.*

Proof: If $ax+b \mod p = ay+b \mod p$ then $a(x-y) \mod p = 0$ and $a \neq 0$ and $(x-y) \neq 0$. However, a and $(x-y)$ cannot divide p since p is prime and $a < p$ and $(x-y) < p$. ■

15.3.4.5 Some Lemmas

Lemma 15.3.7 *If $x \neq y$ then for each (r, s) such that $r \neq s$ and $0 \leq r, s \leq p-1$ there is exactly one a, b such that $ax + b \mod p = r$ and $ay + b \mod p = s$.*

Proof: Solve the two equations:

$$ax + b = r \mod p \text{ and } ay + b = s \mod p$$

We get $a = \frac{r-s}{x-y} \mod p$ and $b = r - ax \mod p$. ■

15.3.4.6 Proof of Claim

Proof: Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

- Let $ax + b \mod p = r$ and $ay + b \mod = s \mod p$.
- Collision if and only if $r = s \mod m$.