Introduction to Randomized Algorithms: QuickSort and QuickSelect

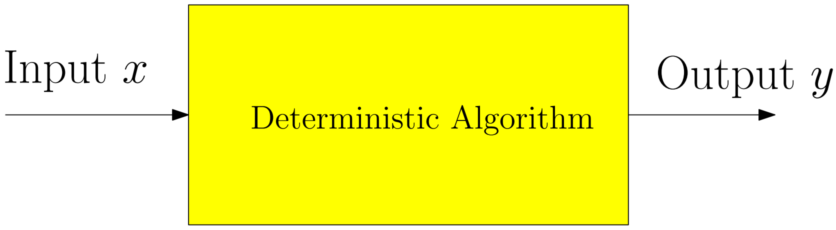
Lecture 13

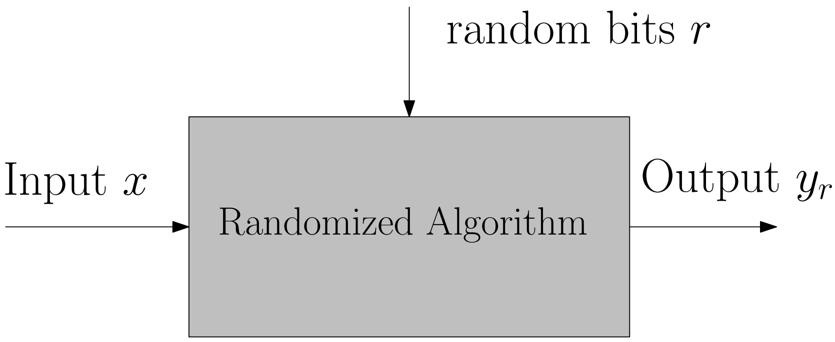
March 8, 2011

Part I

Introduction to Randomized Algorithms

Randomized Algorithms





Example: Randomized **QuickSort**

QuickSort [?]

- Pick a pivot element from array
- Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
- Recursively sort the subarrays, and concatenate them.

Randomized **QuickSort**

- Pick a pivot element *uniformly at random* from the array
- Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
- Recursively sort the subarrays, and concatenate them.

Example: Randomized Quicksort

Recall: **QuickSort** can take $\Omega(n^2)$ time to sort array of size n.

Theorem

Randomized **QuickSort** *sorts a given array of length* n *in* $O(n \log n)$ *expected time.*

Note: On *every* input randomized **QuickSort** takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

Example: Verifying Matrix Multiplication

Problem

Given three $n \times n$ matrices A, B, C is $AB = C$?

Deterministic algorithm:

- Multiply A and B and check if equal to C.
- Running time? $O(n^3)$ by straight forward approach. $O(n^{2.37})$ with fast matrix multiplication (complicated and impractical).

Example: Verifying Matrix Multiplication

Problem

Given three $\mathbf{n} \times \mathbf{n}$ matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is $\mathbf{AB} = \mathbf{C}$?

Randomized algorithm:

- Pick a random $\mathbf{n} \times \mathbf{1}$ vector $\mathbf{r}$.
- Return the answer of the equality $\mathbf{ABr} = \mathbf{Cr}$.
- Running time? $\mathbf{O(n^2)}$!

Theorem

If $\mathbf{AB} = \mathbf{C}$ then the algorithm will always say YES. If $\mathbf{AB} \neq \mathbf{C}$ then the algorithm will say YES with probability at most $\mathbf{1/2}$. Can repeat the algorithm $\mathbf{100}$ times independently to reduce the probability of a false positive to $\mathbf{1/2^{100}}$.

Why randomized algorithms?

- Many many applications in algorithms, data structures and computer science!
- In some cases only known algorithms are randomized or randomness is provably necessary.
- Often randomized algorithms are (much) simpler and/or more efficient.
- Several deep connections to mathematics, physics etc.
- ...
- Lots of fun!

Where do I get random bits?

Question: Are true random bits available in practice?

- Buy them!
- CPUs use physical phenomena to generate random bits.
- Can use pseudo-random bits or semi-random bits from nature. Several fundamental unresolved questions in complexity theory on this topic. Beyond the scope of this course.
- In practice pseudo-random generators work quite well in many applications.
- The model is interesting to think in the abstract and is very useful even as a theoretical construct. One can *derandomize* randomized algorithms to obtain deterministic algorithms.

Average case analysis vs Randomized algorithms

Average case analysis:

- Fix a deterministic algorithm.
- Assume inputs comes from a probability distribution.
- Analyze the algorithm's *average* performance over the distribution over inputs.

Randomized algorithms:

- Algorithm uses random bits in addition to input.
- Analyze algorithms *average* performance over the given input where the average is over the random bits that the algorithm uses.
- On each input behaviour of algorithm is random. Analyze worst-case over all inputs of the (average) performance.

Discrete Probability

We restrict attention to finite probability spaces.

Definition

A discrete probability space is a pair $(\Omega, \Pr)$ consists of finite set Ω of *elementary* events and function $p : \Omega \rightarrow [0, 1]$ which assigns a probability $\Pr[\omega]$ for each $\omega \in \Omega$ such that $\sum_{\omega \in \Omega} \Pr[\omega] = 1$.

Example

An unbiased coin. $\Omega = \{H, T\}$ and $\Pr[H] = \Pr[T] = 1/2$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$.

Discrete Probability

And more examples

Example

A biased coin. $\Omega = \{H, T\}$ and $\Pr[H] = 2/3, \Pr[T] = 1/3$.

Example

Two independent unbiased coins. $\Omega = \{HH, TT, HT, TH\}$ and $\Pr[HH] = \Pr[TT] = \Pr[HT] = \Pr[TH] = 1/4$.

Example

A pair of (highly) correlated dice.
$\Omega = \{(i, j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.
$\Pr[i, i] = 1/6$ for $1 \leq i \leq 6$ and $\Pr[i, j] = 0$ if $i \neq j$.

Events

Definition

Given a probability space $(\Omega, \Pr)$ an **event** is a subset of Ω. In other words an event is a collection of elementary events. The probability of an event A, denoted by $\Pr[A]$, is $\sum_{\omega \in A} \Pr[\omega]$. The complement of an event $A \subseteq \Omega$ is the event $\Omega \setminus A$ frequently denoted by $\bar{A}$.

Events

Examples

Example

A pair of independent dice. $\Omega = \{(i,j) \mid 1 \leq i \leq 6, 1 \leq j \leq 6\}$.

- Let A be the event that the sum of the two numbers on the dice is even. Then $A = \{(i,j) \in \Omega \mid (i+j) \text{ is even}\}$.
  $\Pr[A] = |A|/36 = 1/2$.
- Let B be the event that the first die has 1. Then
  $B = \left\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\right\}$.
  $\Pr[B] = 6/36 = 1/6$.

Independent Events

Definition

Given a probability space $(\Omega, \Pr)$ and two events A, B are **independent** if and only if $\Pr[A \cap B] = \Pr[A]\Pr[B]$. Otherwise they are *dependent*. In other words A, B independent implies one does not affect the other.

Example

Two coins. $\Omega = \{HH, TT, HT, TH\}$ and
$\Pr[HH] = \Pr[TT] = \Pr[HT] = \Pr[TH] = 1/4$.

- A is the event that the first coin is heads and B is the event that second coin is tails. A, B are independent.
- A is the event that the two coins are different. B is the event that the second coin is heads. A, B independent.

Independent Events

Examples

Example

A is the event that both are not tails and B is event that second coin is heads. A, B are dependent.

Random Variables

Definition

Given a probability space $(\Omega, \Pr)$ a (real-valued) random variable X over Ω is a function that maps each elementary event to a real number. In other words $X : \Omega \to \mathbb{R}$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$.

- $X : \Omega \to \mathbb{R}$ where $X(i) = i \mod 2$.
- $Y : \Omega \to \mathbb{R}$ where $Y(i) = i^2$.

Definition

A **binary random variable** is one that takes on values in $\{0, 1\}$.

Indicator Random Variables

Special type of random variables that are quite useful.

Definition

Given a probability space $(\Omega, \Pr)$ and an event $A \subseteq \Omega$ the indicator random variable X_A is a binary random variable where $X_A(\omega) = 1$ if $\omega \in A$ and $X_A(\omega) = 0$ if $\omega \notin A$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$. Let A be the even that i is divisible by 3. Then $X_A(i) = 1$ if $i = 3, 6$ and 0 otherwise.

Expectation

Definition

For a random variable $\mathbf{X}$ over a probability space $(\Omega, \Pr)$ the **expectation** of $\mathbf{X}$ is defined as $\sum_{\omega \in \Omega} \Pr[\omega] \, X(\omega)$. In other words, the expectation is the average value of $\mathbf{X}$ according to the probabilities given by $\Pr[\cdot]$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$.

- $X : \Omega \to \mathbb{R}$ where $X(i) = i \mod 2$. Then $E[X] = 1/2$.
- $Y : \Omega \to \mathbb{R}$ where $Y(i) = i^2$. Then $E[Y] = \sum_{i=1}^{6} \frac{1}{6} \cdot i^2 = 91/6$.

Expectation

Proposition

For an indicator variable X_A, $E[X_A] = \Pr[A]$.

Proof.

$$
\begin{aligned}
E[X_A] &= \sum_{y \in \Omega} X_A(y) \Pr[y] \\
&= \sum_{y \in A} 1 \cdot \Pr[y] + \sum_{y \in \Omega \setminus A} 0 \cdot \Pr[y] \\
&= \sum_{y \in A} \Pr[y] \\
&= \Pr[A] .
\end{aligned}
$$

□

Linearity of Expectation

Lemma

Let $\mathbf{X}, \mathbf{Y}$ *be two random variables over a probability space* $(\Omega, \Pr)$*. Then* $\mathbf{E}[\mathbf{X} + \mathbf{Y}] = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}]$*.*

Proof.

$$\begin{aligned}\mathbf{E}[\mathbf{X}+\mathbf{Y}] &= \sum_{\omega\in\Omega} \Pr[\omega]\left(\mathbf{X}(\omega)+\mathbf{Y}(\omega)\right) \\ &= \sum_{\omega\in\Omega} \Pr[\omega]\,\mathbf{X}(\omega) + \sum_{\omega\in\Omega} \Pr[\omega]\,\mathbf{Y}(\omega) = \mathbf{E}[\mathbf{X}] + \mathbf{E}[\mathbf{Y}]\,.\end{aligned}$$

□

Corollary

$\mathbf{E}[a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + \ldots + a_n\mathbf{X}_n] = \sum_{i=1}^{n} a_i\,\mathbf{E}[\mathbf{X}_i]$.

Types of Randomized Algorithms

Typically one encounters the following types:

- **Las Vegas randomized algorithms:** for a given input x output of algorithm is always correct but the running time is a random variable. In this case we are interested in analyzing the *expected* running time.
- **Monte Carlo randomized algorithms:** for a given input x the running time is deterministic but the output is random; correct with some probability. In this case we are interested in analyzing the *probability* of the correct output (and also the running time).
- Algorithms whose running time and output may both be random.

Analyzing Las Vegas Algorithms

Deterministic algorithm $\mathbf{Q}$ for a problem $\mathbf{\Pi}$:

- Let $\mathbf{Q(x)}$ be the time for $\mathbf{Q}$ to run on input $\mathbf{x}$ of length $|\mathbf{x}|$.
- Worst-case analysis: run time on worst input for a given size $\mathbf{n}$.

$$\mathbf{T_{wc}(n) = \max_{x:|x|=n} Q(x).}$$

Randomized algorithm $\mathbf{R}$ for a problem $\mathbf{\Pi}$:

- Let $\mathbf{R(x)}$ be the time for $\mathbf{Q}$ to run on input $\mathbf{x}$ of length $|\mathbf{x}|$.
- $\mathbf{R(x)}$ is a random variable: depends on random bits used by $\mathbf{R}$.
- $\mathbf{E[R(x)]}$ is the expected running time for $\mathbf{R}$ on $\mathbf{x}$
- Worst-case analysis: expected time on worst input of size $\mathbf{n}$

$$\mathbf{T_{rand-wc}(n) = \max_{x:|x|=n} E[Q(x)]\,.}$$

Analyzing Monte Carlo Algorithms

Randomized algorithm $\mathbf{M}$ for a problem $\mathbf{\Pi}$:

- Let $\mathbf{M(x)}$ be the time for $\mathbf{M}$ to run on input $\mathbf{x}$ of length $|\mathbf{x}|$. For Monte Carlo, assumption is that run time is deterministic.
- Let $\mathbf{Pr[x]}$ be the probability that $\mathbf{M}$ is correct on $\mathbf{x}$.
- $\mathbf{Pr[x]}$ is a random variable: depends on random bits used by $\mathbf{M}$.
- Worst-case analysis: success probability on worst input

$$\mathbf{P_{rand-wc}(n) = \min_{x:|x|=n} Pr[x]\,.}$$

Part II

Randomized Quick Sort and Selection

Randomized **QuickSort**

Randomized **QuickSort**

- Pick a pivot element *uniformly at random* from the array
- Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
- Recursively sort the subarrays, and concatenate them.

Example

- array: 16, 12, 14, 20, 5, 3, 18, 19, 1

Analysis via Recurrence

- Given array $\mathbf{A}$ of size $\mathbf{n}$ let $\mathbf{Q(A)}$ be number of comparisons of randomized **QuickSort** on $\mathbf{A}$.
- Note that $\mathbf{Q(A)}$ is a random variable
- Let $\mathbf{A^i_{\text{left}}}$ and $\mathbf{A^i_{\text{right}}}$ be the left and right arrays obtained if:

pivot is of rank $\mathbf{i}$ in $\mathbf{A}$.

$$\mathbf{Q(A) = n + \sum_{i=1}^{n} Pr[\text{pivot has rank } i] \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}}) \right)}$$

Since each element of $\mathbf{A}$ has probability exactly of $\mathbf{1/n}$ of being chosen:

$$\mathbf{Q(A) = n + \sum_{i=1}^{n} \frac{1}{n} \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}}) \right)}$$

Analysis via Recurrence

Let $\mathbf{T(n) = \max_{A:|A|=n} E[Q(A)]}$ be the worst-case expected running time of randomized **QuickSort** on arrays of size $\mathbf{n}$.

We have, for any $\mathbf{A}$:

$$Q(A) = n + \sum_{i=1}^{n} \Pr[\text{pivot has rank } i] \left(Q(A^i_{\text{left}}) + Q(A^i_{\text{right}}) \right)$$

Therefore, by linearity of expectation:

$$E\Big[Q(A)\Big] = n + \sum_{i=1}^{n} \Pr[\text{pivot of rank } i] \Big(E\big[Q(A^i_{\text{left}})\big] + E\Big[Q(A^i_{\text{right}})\Big] \Big).$$

$$\Rightarrow \quad E\Big[Q(A)\Big] \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i) \right).$$

Analysis via Recurrence

Let $\mathbf{T(n) = \max_{A:|A|=n} E[Q(A)]}$ be the worst-case expected running time of randomized **QuickSort** on arrays of size $\mathbf{n}$.
We derived:

$$E[Q(A)] \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i) \right).$$

Note that above holds for any $\mathbf{A}$ of size $\mathbf{n}$. Therefore

$$\max_{A:|A|=n} E[Q(A)] = T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i) \right).$$

Solving the Recurrence

$$\mathbf{T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n}\left(T(i-1) + T(n-i)\right)}$$

with base case $\mathbf{T(1) = 0}$.

Lemma

$\mathbf{T(n) = O(n \log n)}$.

Proof.

(Guess and) Verify by induction. □

A Slick Analysis of QuickSort

Let $\mathbf{Q(A)}$ be number of comparisons done on input array $\mathbf{A}$:

- For $\mathbf{1 \leq i < j < n}$ let $\mathbf{R_{ij}}$ be the event that rank $\mathbf{i}$ element is compared with rank $\mathbf{j}$ element.
- $\mathbf{X_{ij}}$ is the indicator random variable for $\mathbf{R_{ij}}$. That is, $\mathbf{X_{ij} = 1}$ if rank $\mathbf{i}$ is compared with rank $\mathbf{j}$ element, otherwise $\mathbf{0}$.

$$\mathbf{Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}}$$

and hence by linearity of expectation,

$$\mathbf{E\left[Q(A)\right] = \sum_{1 \leq i < j \leq n} E[X_{ij}] = \sum_{1 \leq i < j \leq n} Pr[R_{ij}]}.$$

A Slick Analysis of **QuickSort**

Question: What is $\mathbf{Pr}[\mathbf{R_{ij}}]$?

Lemma

$\mathbf{Pr}[\mathbf{R_{ij}}] = \frac{2}{(j-i+1)}$.

Proof.

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be elements of $\mathbf{A}$ in sorted order. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

Observation: If pivot is chosen outside $\mathbf{S}$ then all of $\mathbf{S}$ either in left array or right array.

Observation: a_i and a_j separated when a pivot is chosen from $\mathbf{S}$ for the first time. Once separated no comparison.

Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from $\mathbf{S}$ at separation... □

A Slick Analysis of **QuickSort**

Continued...

Lemma

$\mathbf{Pr}[\mathbf{R_{ij}}] = \frac{2}{(j-i+1)}$.

Proof.

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be sort of $\mathbf{A}$. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from $\mathbf{S}$ at separation.

Observation: Given that pivot is chosen from $\mathbf{S}$ the probability that it is a_i or a_j is exactly $2/|S| = 2/(j-i+1)$ since the pivot is chosen uniformly at random from the array. □

A Slick Analysis of **QuickSort**

Continued...

$$\mathbf{E}\left[Q(A)\right] = \sum_{1 \le i < j \le n} \mathbf{E}[X_{ij}] = \sum_{1 \le i < j \le n} \Pr[R_{ij}] .$$

Lemma

$\Pr[R_{ij}] = \frac{2}{(j-i+1)}$.

$$\begin{aligned} \mathbf{E}\left[Q(A)\right] &= \sum_{1 \le i < j \le n} \Pr[R_{ij}] = \sum_{1 \le i < j \le n} \frac{2}{j-i+1} \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} = 2 \sum_{i=1}^{n-1} \sum_{i<j}^{n} \frac{1}{j-i+1} \\ &= 2 \sum_{i=1}^{n-1} (H_{n-i+1} - 1) \le 2 \sum_{1 \le i < n} H_n \\ &\le 2nH_n = O(n \log n) \end{aligned}$$

Randomized Quick Selection

Input Unsorted array A of n integers

Goal Find the jth smallest number in A (*rank* j number)

Randomized Quick Selection

- Pick a pivot element *uniformly at random* from the array
- Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
- Return pivot if rank of pivot is j
- Otherwise recurse on one of the arrays depending on j and their sizes.

Algorithm for Randomized Selection

Assume for simplicity that $\mathbf{A}$ has distinct elements.

```
QuickSelect(A, j):
    Pick pivot x uniformly at random from A
    Partition A into A_less, x, and A_greater using x as pivot
    if (|A_less| = j − 1) then
        return x
    if (|A_less|) ≥ j) then
        return QuickSelect(A_less, j)
    else
        return QuickSelect(A_greater, j − |A_less| − 1)
```


Analysis via Recurrence

- Given array $\mathbf{A}$ of size $\mathbf{n}$ let $\mathbf{Q(A)}$ be number of comparisons of randomized selection on $\mathbf{A}$ for selecting rank $\mathbf{j}$ element.
- Note that $\mathbf{Q(A)}$ is a random variable
- Let $\mathbf{A}^i_{\text{less}}$ and $\mathbf{A}^i_{\text{greater}}$ be the left and right arrays obtained if pivot is rank $\mathbf{i}$ element of $\mathbf{A}$.
- Algorithm recurses on $\mathbf{A}^i_{\text{less}}$ if $\mathbf{j} < \mathbf{i}$ and recurses on $\mathbf{A}^i_{\text{greater}}$ if $\mathbf{j} > \mathbf{i}$ and terminates if $\mathbf{j} = \mathbf{i}$.

$$
\begin{aligned}
\mathbf{Q(A)} &= \mathbf{n} + \sum_{i=1}^{j-1} \Pr[\text{pivot has rank } i]\, \mathbf{Q}(\mathbf{A}^i_{\text{greater}}) \\
&\quad + \sum_{i=j+1}^{n} \Pr[\text{pivot has rank } i]\, \mathbf{Q}(\mathbf{A}^i_{\text{less}})
\end{aligned}
$$

Analyzing the Recurrence

As in **QuickSort** we obtain the following recurrence where $\mathbf{T(n)}$ is the worst-case expected time.

$$T(n) \leq n + \frac{1}{n}\left(\sum_{i=1}^{j-1} T(n-i) + \sum_{i=j}^{n} T(i-1)\right).$$

Theorem

$T(n) = O(n)$.

Proof.

(Guess and) Verify by induction (see next slide). □

Analyzing the recurrence

Theorem

$T(n) = O(n)$.

Prove by induction that $T(n) \leq \alpha n$ for some constant $\alpha \geq 1$ to be fixed later.

Base case: $n = 1$, we have $T(1) = 0$ since no comparisons needed and hence $T(1) \leq \alpha$.

Induction step: Assume $T(k) \leq \alpha k$ for $1 \leq k < n$ and prove it for $T(n)$. We have by the recurrence:

$$\begin{aligned} T(n) &\leq n + \frac{1}{n}\left(\sum_{i=1}^{j-1} T(n-i) + \sum_{i=j^n} T(i-1)\right) \\ &\leq n + \frac{\alpha}{n}\left(\sum_{i=1}^{j-1}(n-i) + \sum_{i=j}^{n}(i-1)\right) \quad \text{by applying induction} \end{aligned}$$

Analyzing the recurrence

$$
\begin{aligned}
T(n) &\le n + \frac{\alpha}{n}\left(\sum_{i=1}^{j-1}(n-i) + \sum_{i=j}^{n}(i-1)\right) \\
&\le n + \frac{\alpha}{n}((j-1)(2n-j)/2 + (n-j+1)(n+j-2)/2) \\
&\le n + \frac{\alpha}{2n}(n^2 + 2nj - 2j^2 - 3n + 4j - 2) \\
&\quad \text{above expression maximized when } j = (n+1)/2\text{: calculus} \\
&\le n + \frac{\alpha}{2n}(3n^2/2 - n) \quad \text{substituting } (n+1)/2 \text{ for } j \\
&\le n + 3\alpha n/4 \\
&\le \alpha n \quad \text{for any constant } \alpha \ge 4
\end{aligned}
$$

Comments on analyzing the recurrence

- Algebra looks messy but intuition suggest that the median is the hardest case and hence can plug $j = n/2$ to simplify without calculus
- Analyzing recurrences comes with practice and after a while one can see things more intuitively

John Von Neumann:
Young man, in mathematics you don't understand things. You just get used to them.