Greedy Algorithms

Lecture 11
March 1, 2011
Part I

Problems and Terminology
Decision Problem: Is the input a YES or NO input?
Example: Given graph $G$, nodes $s, t$, is there a path from $s$ to $t$ in $G$?

Search Problem: Find a solution if input is a YES input.
Example: Given graph $G$, nodes $s, t$, find an $s$-$t$ path.

Optimization Problem: Find a best solution among all solutions for the input.
Example: Given graph $G$, nodes $s, t$, find a shortest $s$-$t$ path.
Terminology

- A problem $\Pi$ consists of an infinite collection of inputs $\{I_1, I_2, \ldots, I\}$. Each input is referred to as an instance.
- The size of an instance $I$ is the number of bits in its representation.
- For an instance $I$, $\text{sol}(I)$ is a set of feasible solutions to $I$. *Typical implicit assumption:* given instance $I$ and $y \in \Sigma^*$, there is a way to check (efficiently!) if $y \in \text{sol}(I)$. In other words, problem is in $\text{NP}$.
- For optimization problems each solution $s \in \text{sol}(I)$ has an associated value. *Typical implicit assumption:* given $s$, can compute value efficiently.
Problem Types

- **Decision Problem**: Given $I$, output whether $\text{sol}(I) = \emptyset$ or not.
- **Search Problem**: Given $I$, find a solution $s \in \text{sol}(I)$ if $\text{sol}(I) \neq \emptyset$.
- **Optimization Problem**: Given $I$,
  - Minimization problem. Find a solution $s \in \text{sol}(I)$ of minimum value
  - Maximization problem. Find a solution $s \in \text{sol}(I)$ of maximum value
- **Notation**: $\text{opt}(I)$: interchangeably (when there is no confusion) used to denote the value of an optimum solution or some fixed optimum solution.
Part II

Greedy Algorithms: Tools and Techniques
What is a Greedy Algorithm?

No real consensus on a universal definition.

Greedy algorithms:

- make decision incrementally in small steps \textit{without backtracking}
- decision at each step is based on improving \textit{local or current} state in a myopic fashion without paying attention to the \textit{global} situation
- decisions often based on some fixed and simple \textit{priority} rules
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- decisions often based on some fixed and simple priority rules
Pros and Cons of Greedy Algorithms

Pros:

- Usually (too) easy to design greedy algorithms
- Easy to implement and often run fast since they are simple
- Several important cases where they are effective/optimal
- Lead to a first-cut heuristic when problem not well understood

Cons:

- **Very often** greedy algorithms don’t work. Easy to lull oneself into believing they work
- Many greedy algorithms possible for a problem and no structured way to find effective ones

CS 473: Every greedy algorithm needs a proof of correctness
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Greedy Algorithm Types

Crude classification:

- **Non-adaptive**: fix some ordering of decisions a priori and stick with the order
- **Adaptive**: make decisions adaptively but greedily/locally at each step

Plan:

- See several examples
- Pick up some proof techniques
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Interval Scheduling

**Input**  A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms)

**Goal**  Schedule as many jobs as possible

- Two jobs with overlapping intervals cannot both be scheduled!
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- Two jobs with overlapping intervals cannot both be scheduled!
$R$ is the set of all requests

$X$ is empty (* $X$ will store all the jobs that will be scheduled *)

while $R$ is not empty do
  choose $i \in R$
  add $i$ to $X$
  remove from $R$ all requests that overlap with $i$
return the set $X$

Main task: Decide the order in which to process requests in $R$
Greedy Template

R is the set of all requests
X is empty (* X will store all the jobs that will be scheduled *)

while R is not empty do
    choose i ∈ R
    add i to X
    remove from R all requests that overlap with i

return the set X

Main task: Decide the order in which to process requests in R
Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.
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Figure: Counter example for earliest start time
Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.

Figure: Counter example for earliest start time
Early Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.

Figure: Counter example for earliest start time
Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.
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Fewest Conflicts

Process jobs in that have the fewest “conflicts” first.
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\[\text{\begin{center}
\begin{tikzpicture}
\draw[ultra thick] (0,0) -- (4,0);
\draw[ultra thick, red] (1,0) -- (3,0);
\end{tikzpicture}
\end{center}}\]
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Earliest Finish Time

Process jobs in the order of their finishing times, beginning with those that finish earliest.
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Optimal Greedy Algorithm

\( R \) is the set of all requests
\( X \) is empty (* \( X \) will store all the jobs that will be scheduled *)

while \( R \) is not empty
    choose \( i \in R \) such that finishing time of \( i \) is least
    add \( i \) to \( X \)
    remove from \( R \) all requests that overlap with \( i \)

return \( X \)

Theorem

The greedy algorithm that picks jobs in the order of their finishing times is optimal.
Correctness: Clearly the algorithm returns a set of jobs that does not have any conflicts.

For a set of requests $R$, let $O$ be an optimal set and let $X$ be the set returned by the greedy algorithm. Then $O = X$? Not likely!

Instead we will show that $|O| = |X|$.
Proving Optimality

- **Correctness:** Clearly the algorithm returns a set of jobs that does not have any conflicts.
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Instead we will show that \(|O| = |X|\)

Sariel (UIUC)
Proof of Optimality: Key Lemma

Lemma

Let \( i_1 \) be first interval picked by Greedy. There exists an optimum solution that contains \( i_1 \).

Proof.

Let \( O \) be an arbitrary optimum solution. If \( i_1 \in O \) we are done.

Claim: If \( i_1 \notin O \) then there is exactly one interval \( j_1 \in O \) that conflicts with \( i_1 \). (proof later)

- Form a new set \( O' \) by removing \( j_1 \) from \( O \) and adding \( i_1 \), that is \( O' = (O - \{j_1\}) \cup \{i_1\} \).
- From claim, \( O' \) is a feasible solution (no conflicts).
- Since \( |O'| = |O| \), \( O' \) is also an optimum solution and it contains \( i_1 \).
Proof of Optimality: Key Lemma

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- Since $|O'| = |O|$, $O'$ is also an optimum solution and it contains $i_1$. 

Proof of Claim:

Let $i = i_1$ be the first interval picked by Greedy. There exists an optimum solution that contains $i$.

Proof.

Let $O$ be an arbitrary optimum solution. If $i \in O$ we are done.

Claim: If $i \notin O$ then there is exactly one interval $j \in O$ that conflicts with $i$. (proof later)

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Proof of Optimality: Key Lemma

**Lemma**

Let $i_1$ be first interval picked by Greedy. There exists an optimum solution that contains $i_1$.

**Proof.**

Let $O$ be an arbitrary optimum solution. If $i_1 \in O$ we are done. **Claim:** If $i_1 \not\in O$ then there is exactly one interval $j_1 \in O$ that conflicts with $i_1$. (proof later)

- Form a new set $O'$ by removing $j_1$ from $O$ and adding $i_1$, that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
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Claim

If $i_1 \notin O$ then there is exactly one interval $j_1 \in O$ that conflicts with $i_1$.

Proof.

- Suppose $j_1, j_2 \in O$ such that $j_1 \neq j_2$ and both $j_1$ and $j_2$ conflict with $i_1$.
- Since $i_1$ has earliest finish time, $j_1$ and $i_1$ overlap at $f(i_1)$.
- For same reason $j_2$ also overlaps with $i_1$ at $f(i_1)$.
- Implies that $j_1, j_2$ overlap at $f(i_1)$ contradicting the feasibility of $O$.

See figure in next slide.
Figure: Since $i_1$ has the earliest finish time, any interval that conflicts with it does so at $f(i_1)$. This implies $j_1$ and $j_2$ conflict.
Proof by Induction on number of intervals.

**Base Case:** \( n = 1 \). Trivial since Greedy picks one interval.

**Induction Step:** Assume theorem holds for \( i < n \).

Let \( I \) be an instance with \( n \) intervals

\( I' \): \( I \) with \( i_1 \) and all intervals that overlap with \( i_1 \) removed

**G(I), G(I')**: Solution produced by Greedy on \( I \) and \( I' \)

From Lemma, there is an optimum solution \( O \) to \( I \) and \( i_1 \in O \).

Let \( O' = O - \{i_1\} \). \( O' \) is a solution to \( I' \).

\[
|G(I)| = 1 + |G(I')| \quad \text{(from Greedy description)}
\]
\[
\geq 1 + |O'| \quad \text{(By induction, G(I') is optimum for I')} \\
= |O|
\]
Implementation and Running Time

Initially $R$ is the set of all requests

$X$ is empty (* $X$ will store all the jobs that will be scheduled *)

while $R$ is not empty

choose $i \in R$ such that finishing time of $i$ is least

if $i$ does not overlap with requests in $X$

add $i$ to $X$

remove $i$ from $R$

return the set $X$

- Presort all requests based on finishing time. $O(n \log n)$ time
- Now choosing least finishing time is $O(1)$
- Keep track of the finishing time of the last request added to $A$. Then check if starting time of $i$ later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$
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- Thus, checking non-overlapping is \( O(1) \)
- Total time \( O(n \log n + n) = O(n \log n) \)
Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.

Instead of maximizing the total number of requests, associate value/weight with each job that is scheduled. Try to schedule jobs to maximize total value/weight. No greedy algorithm. Will be seen later in this course to illustrate dynamic programming.

All requests need not be known at the beginning. Such online algorithms are a subject of research.
Scheduling all Requests

**Input** A set of lectures, with start and end times

**Goal** Find the minimum number of classrooms, needed to schedule all the lectures such two lectures do not occur at the same time in the same room.

---

**Figure:** A schedule requiring 4 classrooms

**Figure:** A schedule requiring 3 classrooms
Scheduling all Requests

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**Figure:** A schedule requiring 4 classrooms

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Greedy Algorithm

Initially \( R \) is the set of all requests
\[
d = 0 \quad (* \text{number of classrooms} * )
\]

while \( R \) is not empty
  choose \( i \in R \) such that start time of \( i \) is earliest
  if \( i \) can be scheduled in some class-room \( k \leq d \)
    schedule lecture \( i \) in class-room \( k \)
  else
    allocate a new class-room \( d + 1 \) and schedule lecture \( i \) in \( d + 1 \)
    \( d = d + 1 \)

What order should we process requests in? According to start times (breaking ties arbitrary)
Greedy Algorithm

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What order should we process requests in? According to start times (breaking ties arbitrarily)
Depth of Lectures

Definition

- For a set of lectures $R$, $k$ are said to be in conflict if there is some time $t$ such that there are $k$ lectures going on at time $t$.
- The depth of a set of lectures $R$ is the maximum number of lectures in conflict at any time.
**Definition**

- For a set of lectures $R$, $k$ are said to be in **conflict** if there is some time $t$ such that there are $k$ lectures going on at time $t$.

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**Depth of Lectures**

**Definition**
- For a set of lectures $R$, $k$ are said to be in conflict if there is some time $t$ such that there are $k$ lectures going on at time $t$.
- The depth of a set of lectures $R$ is the maximum number of lectures in conflict at any time.

![Diagram with lectures c, d, f, j, b, g, i, a, e, h in conflict at various times]
Lemma

For any set \( R \) of lectures, the number of class-rooms required is at least the depth of \( R \).

Proof.

All lectures that are in conflict must be scheduled in different rooms.
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Proof.

All lectures that are in conflict must be scheduled in different rooms.
Lemma

Let $d$ be the depth of the set of lectures $R$. The number of class-rooms used by the greedy algorithm is $d$.

Proof.

- Suppose the greedy algorithm uses more than $d$ rooms. Let $j$ be the first lecture that is scheduled in room $d + 1$.
- Since we process lectures according to start times, there are $d$ lectures that start (at or) before $j$ and which are in conflict with $j$.
- Thus, at the start time of $j$, there are at least $d + 1$ lectures in conflict, which contradicts the fact that the depth is $d$. 

\end{proof}
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- Thus, at the start time of $j$, there are at least $d + 1$ lectures in conflict, which contradicts the fact that the depth is $d$. \qed
Figure

\[ s(j) \]

\[ j \]

no such job scheduled before \( j \)
Correctness

Observation

The greedy algorithm does not schedule two overlapping lectures in the same room.

Theorem

The greedy algorithm is correct and uses the optimal number of class-rooms.
Initially $R$ is the set of all requests
$d = 0$ (* number of classrooms *)

while $R$ is not empty

choose $i \in R$ such that start time of $i$ is earliest

if $i$ can be scheduled in some class-room $k \leq d$

schedule lecture $i$ in class-room $k$

else

allocate a new class-room $d + 1$ and schedule lecture $i$ in $d + 1$

$d = d + 1$

Presort according to start times. Picking lecture with earliest
start time can be done in $O(1)$ time.

Keep track of the finish time of last lecture in each room.

Total time
Initially \( R \) is the set of all requests

\[
\begin{align*}
  d &= 0 \quad (* \text{number of classrooms} *) \\
  \text{while } R \text{ is not empty} \\
  &\quad \text{choose } i \in R \text{ such that start time of } i \text{ is earliest} \\
  &\quad \text{if } i \text{ can be scheduled in some classroom } k \leq d \\
  &\quad \quad \text{schedule lecture } i \text{ in classroom } k \\
  &\quad \text{else} \\
  &\quad \quad \text{allocate a new classroom } d + 1 \text{ and schedule lecture } i \text{ in } d + 1 \\
  &\quad d = d + 1
\end{align*}
\]

- Presort according to start times. Picking lecture with earliest start time can be done in \( O(1) \) time.
- Keep track of the finish time of last lecture in each room.

- Total time
Initially $R$ is the set of all requests

$d = 0$ (* number of classrooms *)

While $R$ is not empty

1. Choose $i \in R$ such that start time of $i$ is earliest
2. If $i$ can be scheduled in some classroom $k \leq d$
   - Schedule lecture $i$ in classroom $k$
3. Else
   - Allocate a new classroom $d + 1$ and schedule lecture $i$ in $d + 1$
   - $d = d + 1$

- Presort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.
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Presort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.

Keep track of the finish time of last lecture in each room.

Checking conflict takes $O(d)$ time.

Total time $= O(n \log n + nd)$
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        schedule lecture $i$ in class-room $k$
    else
        allocate a new class-room $d + 1$ and schedule lecture $i$ in $d + 1$
        $d = d + 1$
• Presort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.
• Keep track of the finish time of last lecture in each room.
• With priority queues, checking conflict takes $O(\log d)$ time.
• Total time $= O(n \log n + n \log d) = O(n \log n)$
Scheduling to Minimize Lateness

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job \( i \) starts at time \( s_i \) then it will finish at time \( f_i = s_i + t_i \), where \( t_i \) is its processing time. \( d_i \): deadline.
- The lateness of a job is \( l_i = \max(0, f_i - d_i) \).
- Schedule all jobs such that \( L = \max l_i \) is minimized.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_i )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( d_i )</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

\[ l_1 = 2 \quad l_5 = 0 \quad l_4 = 6 \]
Scheduling to Minimize Lateness

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job $i$ starts at time $s_i$ then it will finish at time $f_i = s_i + t_i$, where $t_i$ is its processing time. $d_i$: deadline.
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$l_1 = 2$, $l_5 = 0$, $l_4 = 6$
A Simpler Feasibility Problem

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job $i$ starts at time $s_i$ then it will finish at time $f_i = s_i + t_i$, where $t_i$ is its processing time.
- Schedule all jobs such that each of them completes before its deadline (in other words $L = \max_i l_i = 0$).

Definition

A schedule is **feasible** if all jobs finish before their deadline.
Initially $R$ is the set of all requests

\[ \text{curr\_time} = 0 \]

while $R$ is not empty do

choose $i \in R$

\[ \text{curr\_time} = \text{curr\_time} + t_i \]

if (curr\_time $> d_i$) then

return ‘‘no feasible schedule’’

return ‘‘found feasible schedule’’

Main task: Decide the order in which to process jobs in $R$
Greedy Template

Initially $R$ is the set of all requests

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Three Algorithms

- Shortest job first — sort according to $t_i$.
- Shortest slack first — sort according to $d_i - t_i$.
- **EDF** = Earliest deadline first — sort according to $d_i$.

Counter examples for first two: exercise
Three Algorithms

- Shortest job first — sort according to $t_i$.
- Shortest slack first — sort according to $d_i - t_i$.
- EDF = Earliest deadline first — sort according to $d_i$.

Counter examples for first two: exercise
Theorem

*Greedy with EDF rule for picking requests correctly decides if there is a feasible schedule.*

Proof via an exchange argument.

Idle time: time during which machine is not working.

Lemma

*If there is a feasible schedule then there is one with no idle time before all jobs are finished.*
Earliest Deadline First

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Inversions

Definition
A schedule $S$ is said to have an inversion if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$.

Claim
If a schedule $S$ has an inversion then there is an inversion between two adjacently scheduled jobs.

Proof: exercise.
Inversions

Definition

A schedule $S$ is said to have an inversion if there are jobs $i$ and $j$ such that $S$ schedules $i$ before $j$, but $d_i > d_j$.

Claim

If a schedule $S$ has an inversion then there is an inversion between two adjacently scheduled jobs.

Proof: exercise.
Lemma

If there is a feasible schedule, then there is one with no inversions.

Proof Sketch.

Let $S$ be a schedule with minimum number of inversions.

- If $S$ has 0 inversions, done.
- Suppose $S$ has one or more inversions. By claim there are two adjacent jobs $i$ and $j$ that define an inversion.
- Swap positions of $i$ and $j$.
- New schedule is still feasible. (Why?)
- New schedule has one fewer inversion — contradiction!
Back to Minimizing Lateness

Goal: schedule to minimize $L = \max_i l_i$.

How can we use algorithm for simpler feasibility problem?

Given a lateness bound $L$, can we check if there is a schedule such that $\max_i l_i \leq L$?

Yes! Set $d'_i = d_i + L$ for each job $i$. Use feasibility algorithm with new deadlines.

How can we find minimum $L$? Binary search!
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Yes! Set \( d'_i = d_i + L \) for each job \( i \). Use feasibility algorithm with new deadlines.

How can we find \textit{minimum} \( L \)? Binary search!
Binary search for finding minimum lateness

\[ L = L_{\text{min}} = 0 \]
\[ L_{\text{max}} = \sum_i t_i \quad \text{// why is this sufficient?} \]

While \( L_{\text{min}} < L_{\text{max}} \) do
  \[ L = \left\lfloor \frac{L_{\text{max}} + L_{\text{min}}}{2} \right\rfloor \]
  check if there is a feasible schedule with lateness \( L \)
  if ‘‘yes’’ then \( L_{\text{max}} = L \)
  else \( L_{\text{min}} = L + 1 \)
end while

return \( L \)

Running time: \( O(n \log n \cdot \log T) \) where \( T = \sum_i t_i \)
- \( O(n \log n) \) for feasibility test (sort by deadlines)
- \( O(\log T) \) calls to feasibility test in binary search
Binary search for finding minimum lateness

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While \( L_{\text{min}} < L_{\text{max}} \) do
  \[ L = \lfloor (L_{\text{max}} + L_{\text{min}})/2 \rfloor \]
  check if there is a feasible schedule with lateness \( L \)
  if ‘‘yes’’ then \( L_{\text{max}} = L \)
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- \( O(n \log n) \) for feasibility test (sort by deadlines)
- \( O(\log T) \) calls to feasibility test in binary search
Do we need binary search?

What happens in each call?

EDF algorithm with deadlines \( d_i' = d_i + L \).

Greedy with EDF schedules the jobs in the same order for all \( L \)!

Maybe there is a direct greedy algorithm for minimizing maximum lateness?
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Greedy with **EDF** schedules the jobs in the same order for all \(L\)!!!

Maybe there is a direct greedy algorithm for minimizing maximum lateness?
Greedy Algorithm for Minimizing Lateness

Initially \( R \) is the set of all requests

\[
\begin{align*}
\text{curr\_time} &= 0 \\
\text{curr\_late} &= 0 \\
\text{while } R \text{ is not empty} & \quad \text{choose } i \in R \text{ with earliest deadline} \\
\quad & \quad \text{curr\_time} = \text{curr\_time} + t_i \\
\quad & \quad \text{late} = \text{curr\_time} - d_i \\
\quad & \quad \text{curr\_late} = \max(\text{late}, \text{curr\_late}) \\
\text{return } \text{curr\_late}
\end{align*}
\]

Exercise: argue directly that above algorithm is correct (see book).

Can be easily implemented in \( O(n \log n) \) time after sorting jobs.
Greedy Algorithm for Minimizing Lateness

Initially $R$ is the set of all requests

curr_time = 0
curr_late = 0

while $R$ is not empty
    choose $i \in R$ with earliest deadline
    curr_time = curr_time + $t_i$
    late = curr_time − $d_i$
    curr_late = max(late, curr_late)

return curr_late

Exercise: argue directly that above algorithm is correct (see book).

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Greedy Algorithm for Minimizing Lateness

Initially $R$ is the set of all requests

$\text{curr\_time} = 0$

$\text{curr\_late} = 0$

while $R$ is not empty

\begin{align*}
\text{choose } i \in R \text{ with earliest deadline} \\
\text{curr\_time} &= \text{curr\_time} + t_i \\
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\end{align*}

return $\text{curr\_late}$

Exercise: argue directly that above algorithm is correct (see book).

Can be easily implemented in $O(n \log n)$ time after sorting jobs.
Greedy Analysis: Overview

- **Greedy’s first step leads to an optimum solution.** Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.

- **Greedy algorithm stays ahead.** Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.

- **Structural property of solution.** Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning.

- **Exchange argument.** Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.
Takeaway Points

- Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.

- *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.

- Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.