CS 473: Fundamental Algorithms, Spring 2011

Greedy Algorithms

Lecture 11
March 1, 2011

Part I

Problems and Terminology

Problem Types

- Decision Problem: Is the input a YES or NO input?
  Example: Given graph $\mathbf{G}$, nodes $\mathbf{s}, \mathbf{t}$, is there a path from $\mathbf{s}$ to $\mathbf{t}$ in $\mathbf{G}$?
- Search Problem: Find a *solution* if input is a YES input.
  Example: Given graph $\mathbf{G}$, nodes $\mathbf{s}, \mathbf{t}$, find an $\mathbf{s}$-$\mathbf{t}$ path.
- Optimization Problem: Find a *best* solution among all solutions for the input.
  Example: Given graph $\mathbf{G}$, nodes $\mathbf{s}, \mathbf{t}$, find a shortest $\mathbf{s}$-$\mathbf{t}$ path.

Terminology

- A problem $\mathbf{\Pi}$ consists of an *infinite* collection of inputs $\{\mathbf{I_1}, \mathbf{I_2}, \ldots, \}$. Each input is referred to as an instance.
- The size of an instance $\mathbf{I}$ is the number of bits in its representation.
- For an instance $\mathbf{I}$, $\mathbf{sol(I)}$ is a set of feasible solutions to $\mathbf{I}$.
  Typical implicit assumption: given instance $\mathbf{I}$ and $\mathbf{y} \in \mathbf{\Sigma}^*$, there is a way to check (efficiently!) if $\mathbf{y} \in \mathbf{sol(I)}$. In other words, problem is in **NP**.
- For optimization problems each solution $\mathbf{s} \in \mathbf{sol(I)}$ has an associated value. *Typical implicit assumption:* given $\mathbf{s}$, can compute value efficiently.

Problem Types

- Decision Problem: Given $\mathbf{I}$ output whether $\mathbf{sol(I)} = \emptyset$ or not.
- Search Problem: Given $\mathbf{I}$, find a solution $\mathbf{s} \in \mathbf{sol(I)}$ if $\mathbf{sol(I)} \neq \emptyset$.
- Optimization Problem: Given $\mathbf{I}$,
  - Minimization problem. Find a solution $\mathbf{s} \in \mathbf{sol(I)}$ of minimum value
  - Maximization problem. Find a solution $\mathbf{s} \in \mathbf{sol(I)}$ of maximum value
  - Notation: $\mathbf{opt(I)}$: interchangeably (when there is no confusion) used to denote the value of an optimum solution or some fixed optimum solution.

Part II

Greedy Algorithms: Tools and Techniques

What is a Greedy Algorithm?

No real consensus on a universal definition.

Greedy algorithms:

- make decision incrementally in small steps *without backtracking*
- decision at each step is based on improving *local or current* state in a myopic fashion without paying attention to the *global* situation
- decisions often based on some fixed and simple *priority* rules

Pros and Cons of Greedy Algorithms

Pros:

- Usually (too) easy to design greedy algorithms
- Easy to implement and often run fast since they are simple
- Several important cases where they are effective/optimal
- Lead to a first-cut heuristic when problem not well understood

Cons:

- **Very often** greedy algorithms don't work. Easy to lull oneself into believing they work
- Many greedy algorithms possible for a problem and no structured way to find effective ones

CS 473: Every greedy algorithm needs a proof of correctness

Greedy Algorithm Types

Crude classification:

- Non-adaptive: fix some ordering of decisions a priori and stick with the order
- Adaptive: make decisions adaptively but greedily/locally at each step

Plan:

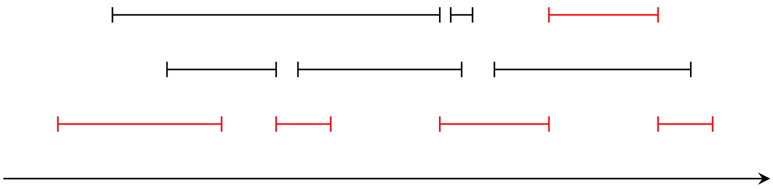
- See several examples
- Pick up some proof techniques

Interval Scheduling

Input A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms)

Goal Schedule as many jobs as possible

- Two jobs with overlapping intervals cannot both be scheduled!



Greedy Template

```
$R$ is the set of all requests
$X$ is empty (* $X$ will store all the jobs that will be scheduled *)
while $R$ is not empty do
    choose $i \in R$
    add $i$ to $X$
    remove from $R$ all requests that overlap with $i$
return the set $X$
```

Main task: Decide the order in which to process requests in R

ES SP FC EF

Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.

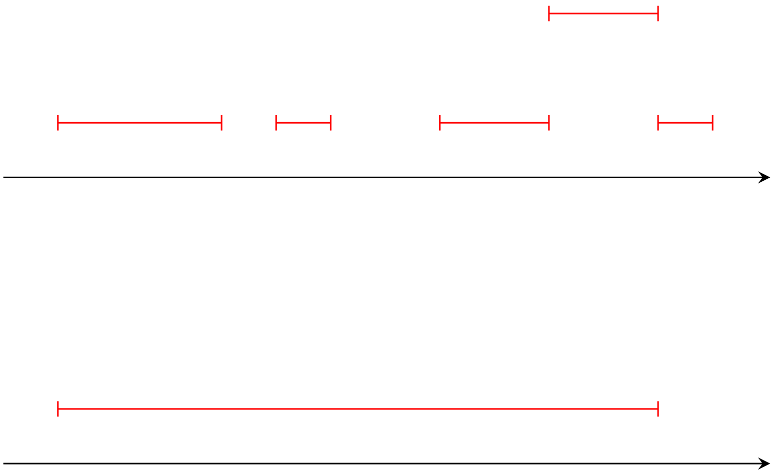


Figure: Counter example for earliest start time

Back Counter

Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.

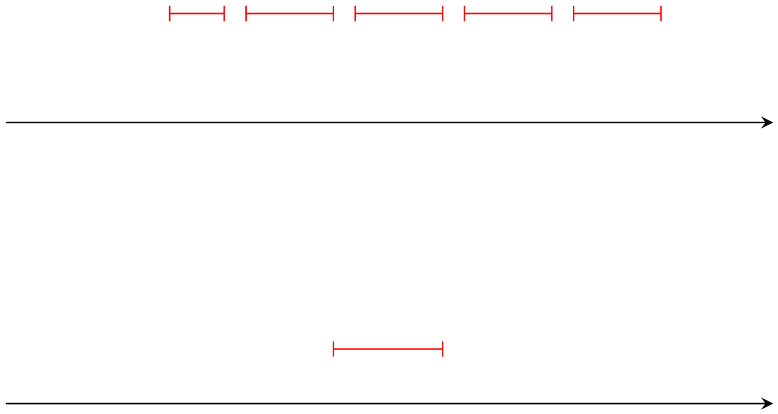


Figure: Counter example for smallest processing time

Fewest Conflicts

Process jobs in that have the fewest "conflicts" first.

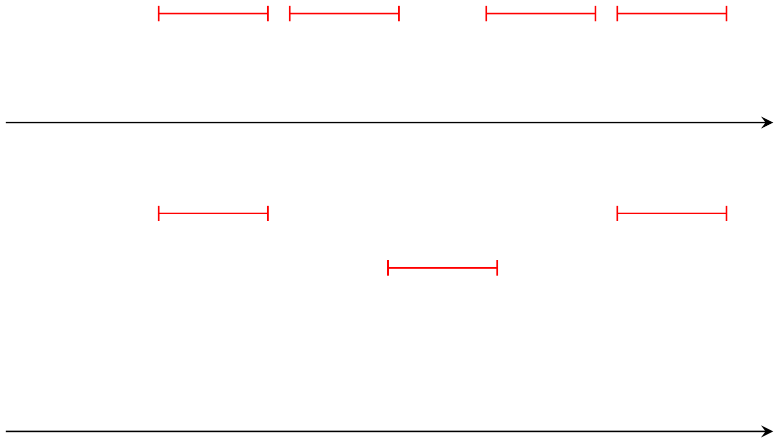
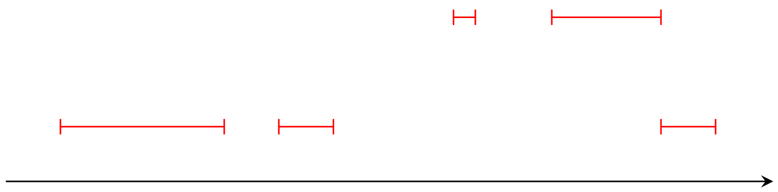


Figure: Counter example for fewest conflicts

Earliest Finish Time

Process jobs in the order of their finishing times, beginning with those that finish earliest.



Optimal Greedy Algorithm

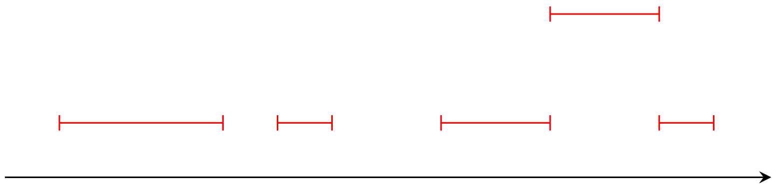
```
R is the set of all requests
X is empty (* X will store all the jobs that will be scheduled *)
while R is not empty
    choose $i \in R$ such that finishing time of i is least
    add i to X
    remove from R all requests that overlap with i
return X
```

Theorem

The greedy algorithm that picks jobs in the order of their finishing times is optimal.

Proving Optimality

- Correctness: Clearly the algorithm returns a set of jobs that does not have any conflicts
- For a set of requests $\mathbf{R}$, let $\mathbf{O}$ be an optimal set and let $\mathbf{X}$ be the set returned by the greedy algorithm. Then $\mathbf{O} = \mathbf{X}$? Not likely!



Instead we will show that $|\mathbf{O}| = |\mathbf{X}|$

Proof of Optimality: Key Lemma

Lemma

Let $\mathbf{i}_1$ be first interval picked by Greedy. There exists an optimum solution that contains $\mathbf{i}_1$.

Proof.

Let $\mathbf{O}$ be an *arbitrary* optimum solution. If $\mathbf{i}_1 \in \mathbf{O}$ we are done.
Claim: If $\mathbf{i}_1 \not\in \mathbf{O}$ then there is exactly one interval $\mathbf{j}_1 \in \mathbf{O}$ that conflicts with $\mathbf{i}_1$. (proof later)

- Form a new set $\mathbf{O}'$ by removing $\mathbf{j}_1$ from $\mathbf{O}$ and adding $\mathbf{i}_1$, that is $\mathbf{O}' = (\mathbf{O} - \{\mathbf{j}_1\}) \cup \{\mathbf{i}_1\}$.
- From claim, $\mathbf{O}'$ is a *feasible* solution (no conflicts).
- Since $|\mathbf{O}'| = |\mathbf{O}|$, $\mathbf{O}'$ is also an optimum solution and it contains $\mathbf{i}_1$. ☐

Proof of Claim

Claim

If $\mathbf{i_1} \not\in \mathbf{O}$ *then there is exactly one interval* $\mathbf{j_1} \in \mathbf{O}$ *that conflicts with* $\mathbf{i_1}$.

Proof.

- Suppose $\mathbf{j_1}, \mathbf{j_2} \in \mathbf{O}$ such that $\mathbf{j_1} \neq \mathbf{j_2}$ and both $\mathbf{j_1}$ and $\mathbf{j_2}$ conflict with $\mathbf{i_1}$.
- Since $\mathbf{i_1}$ has earliest finish time, $\mathbf{j_1}$ and $\mathbf{i_1}$ overlap at $\mathbf{f(i_1)}$.
- For same reason $\mathbf{j_2}$ also overlaps with $\mathbf{i_1}$ at $\mathbf{f(i_1)}$.
- Implies that $\mathbf{j_1}, \mathbf{j_2}$ overlap at $\mathbf{f(i_1)}$ contradicting the feasibility of $\mathbf{O}$.

See figure in next slide. $\square$

Figure for proof of Claim

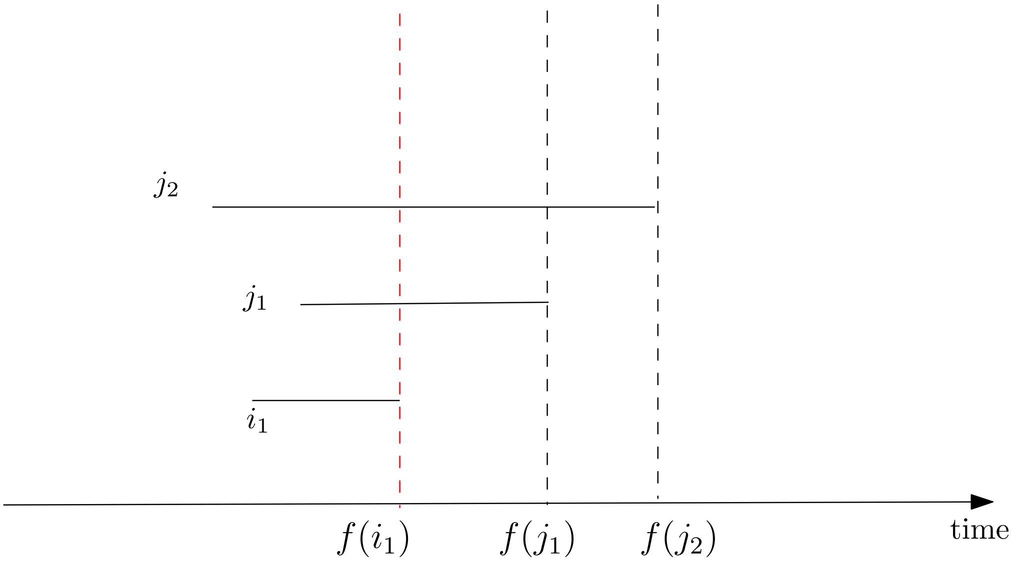


Figure: Since $\mathbf{i_1}$ has the earliest finish time, any interval that conflicts with it does so at $\mathbf{f(i_1)}$. This implies $\mathbf{j_1}$ and $\mathbf{j_2}$ conflict.

Proof of Optimality of Earliest Finish Time First

Proof by Induction on number of intervals.

Base Case: $n = 1$. Trivial since Greedy picks one interval.
Induction Step: Assume theorem holds for $i < n$.
Let I be an instance with n intervals
I': I with i_1 and all intervals that overlap with i_1 removed
$G(I), G(I')$: Solution produced by Greedy on I and I'
From Lemma, there is an optimum solution O to I and $i_1 \in O$.
Let $O' = O - \{i_1\}$. O' is a solution to I'.

$$\begin{aligned} |G(I)| &= 1 + |G(I')| && \text{(from Greedy description)} \\ &\geq 1 + |O'| && \text{(By induction, } G(I') \text{ is optimum for } I'\text{)} \\ &= |O| \end{aligned}$$

□

Implementation and Running Time

```
Initially $R$ is the set of all requests
$X$ is empty (* $X$ will store all the jobs that will be scheduled *)
while $R$ is not empty
    choose $i \in R$ such that finishing time of i is least
    if $i$ does not overlap with requests in $X$
        add $i$ to $X$
    remove $i$ from $R$
return the set $X$
```

- Presort all requests based on finishing time. $O(n \log n)$ time
- Now choosing least finishing time is $O(1)$
- Keep track of the finishing time of the last request added to A. Then check if starting time of i later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$

Comments

- Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
- Instead of maximizing the total number of requests, associate *value/weight* with each job that is scheduled. Try to schedule jobs to maximize total value/weight. No greedy algorithm. Will be seen later in this course to illustrate dynamic programming.
- All requests need not be known at the beginning. Such *online* algorithms are a subject of research

---

Scheduling all Requests

Input A set of lectures, with start and end times

Goal Find the minimum number of classrooms, needed to schedule all the lectures such two lectures do not occur at the same time in the same room.

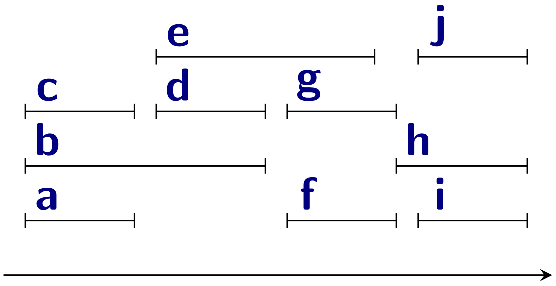


Figure: A schedule requiring 4 classrooms

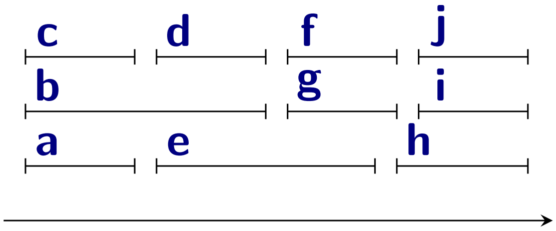


Figure: A schedule requiring 3 classrooms

Greedy Algorithm

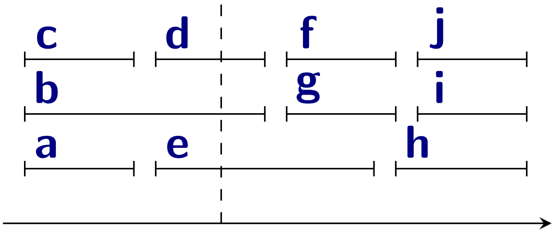
```
Initially $R$ is the set of all requests
$d = 0$ (* number of classrooms *)
while $R$ is not empty
    choose $i \in R$ such that start time of $i$ is earliest
    if $i$ can be scheduled in some class-room $k \leq d$
        schedule lecture $i$ in class-room $k$
    else
        allocate a new class-room $d + 1$ and schedule lecture $i$ in $d + 1$
        $d = d + 1$
```

What order should we process requests in? According to times (breaking ties arbitrarily)

Depth of Lectures

Definition

- For a set of lectures R, k are said to be in conflict if there is some time t such that there are k lectures going on at time t.
- The depth of a set of lectures R is the maximum number of lectures in conflict at any time.



Depth and Number of Class-rooms

Lemma

For any set $\mathbf{R}$ *of lectures, the number of class-rooms required is at least the depth of* $\mathbf{R}$*.*

Proof.

All lectures that are in conflict must be scheduled in different rooms. ☐

Number of Class-rooms used by Greedy Algorithm

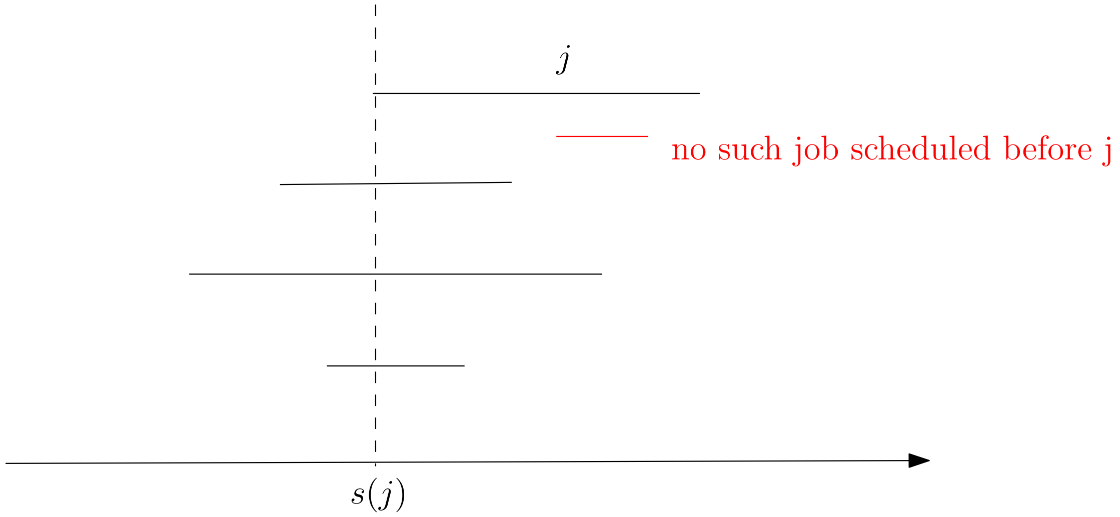
Lemma

Let $\mathbf{d}$ *be the depth of the set of lectures* $\mathbf{R}$*. The number of class-rooms used by the greedy algorithm is* $\mathbf{d}$*.*

Proof.

- Suppose the greedy algorithm uses more that $\mathbf{d}$ rooms. Let $\mathbf{j}$ be the first lecture that is scheduled in room $\mathbf{d+1}$.
- Since we process lectures according to start times, there are $\mathbf{d}$ lectures that start (at or) before $\mathbf{j}$ and which are in conflict with $\mathbf{j}$.
- Thus, *at the start time of* $\mathbf{j}$, there are at least $\mathbf{d+1}$ lectures in conflict, which contradicts the fact that the depth is $\mathbf{d}$. ☐

Figure



Correctness

Observation

The greedy algorithm does not schedule two overlapping lectures in the same room.

Theorem

The greedy algorithm is correct and uses the optimal number of class-rooms.

Implementation and Running Time

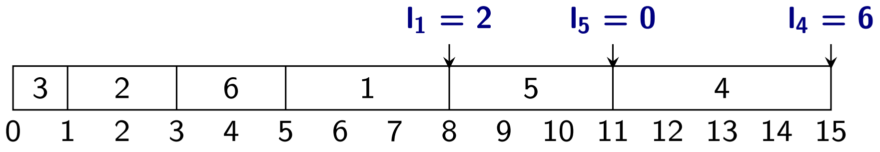
```
Initially $R$ is the set of all requests
$d = 0$ (* number of classrooms *)
while $R$ is not empty
    choose $i \in R$ such that start time of $i$ is earliest
    if $i$ can be scheduled in some class-room $k \le d$
        schedule lecture $i$ in class-room $k$
    else
        allocate a new class-room $d + 1$ and schedule lecture $i$ in $d + 1$
        $d = d + 1$
```

- Presort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.
- Keep track of the finish time of last lecture in each room.
- Checking conflict takes $O(d)$ time.With priority queues, checking conflict takes $O(\log d)$ time.
- Total time $= O(n \log n + nd) = O(n \log n + n \log d) = O(n \log n)$

Scheduling to Minimize Lateness

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job i starts at time s_i then it will finish at time $f_i = s_i + t_i$, where t_i is its processing time. d_i: deadline.
- The lateness of a job is $l_i = \max(0, f_i - d_i)$.
- Schedule all jobs such that $L = \max l_i$ is minimized.

	1	2	3	4	5	6
t_i	3	2	1	4	3	2
d_i	6	8	9	9	14	15



A Simpler Feasibility Problem

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job $\mathbf{i}$ starts at time $\mathbf{s_i}$ then it will finish at time $\mathbf{f_i = s_i + t_i}$, where $\mathbf{t_i}$ is its processing time.
- Schedule all jobs such that each of them completes before its deadline (in other words $\mathbf{L = \max_i l_i = 0}$).

Definition

A schedule is feasible if all jobs finish before their deadline.

Greedy Template

```
Initially $R$ is the set of all requests
$curr\_time = 0$
while $R$ is not empty do
    choose $i \in R$
    $curr\_time = curr\_time + t_i$
    if ($curr\_time > d_i$) then
        return ``no feasible schedule''

return ``found feasible schedule''
```

Main task: Decide the order in which to process jobs in $\mathbf{R}$

Three Algorithms

- Shortest job first — sort according to $\mathbf{t_i}$.
- Shortest slack first — sort according to $\mathbf{d_i - t_i}$.
- EDF = Earliest deadline first — sort according to $\mathbf{d_i}$.

Counter examples for first two: exercise

Earliest Deadline First

Theorem

Greedy with EDF *rule for picking requests correctly decides if there is a feasible schedule.*

Proof via an exchange argument.

Idle time: time during which machine is not working.

Lemma

If there is a feasible schedule then there is one with no idle time before all jobs are finished.

Inversions

Definition

A schedule **S** is said to have an inversion if there are jobs **i** and **j** such that **S** schedules **i** before **j**, but $\mathbf{d_i} > \mathbf{d_j}$.

Claim

If a schedule **S** *has an inversion then there is an inversion between two* adjacently *scheduled jobs.*

Proof: exercise.

Main Lemma

Lemma

If there is a feasible schedule, then there is one with no inversions.

Proof Sketch.

Let **S** be a schedule with minimum number of inversions.

- If **S** has **0** inversions, done.
- Suppose **S** has one or more inversions. By claim there are two adjacent jobs **i** and **j** that define an inversion.
- Swap positions of **i** and **j**.
- New schedule is still feasible. (Why?)
- New schedule has one fewer inversion — contradiction!

☐

Back to Minimizing Lateness

Goal: schedule to minimize $\mathbf{L} = \max_i l_i$.

How can we use algorithm for simpler feasibility problem?

Given a lateness bound $\mathbf{L}$, can we check if there is a schedule such that $\max_i l_i \leq \mathbf{L}$?

Yes! Set $d_i' = d_i + L$ for each job $\mathbf{i}$. Use feasibility algorithm with new deadlines.

How can we find *minimum* $\mathbf{L}$? Binary search!

Binary search for finding minimum lateness

```
$L = L_{min} = 0$
$L_{max} = \sum_i t_i$ // why is this sufficient?
While $L_{min} < L_{max}$ do
    $L = \lfloor (L_{max} + L_{min})/2 \rfloor$
    check if there is a feasible schedule with lateness $L$
    if ``yes'' then $L_{max} = L$
    else $L_{min} = L + 1$
end while
return $L$
```

Running time: $O(n \log n \cdot \log T)$ where $T = \sum_i t_i$

- $O(n \log n)$ for feasibility test (sort by deadlines)
- $O(\log T)$ calls to feasibility test in binary search

Do we need binary search?

What happens in each call?
EDF algorithm with deadlines $\mathbf{d'_i = d_i + L}$.

Greedy with EDF schedules the jobs in the same order for all $\mathbf{L}$!!!

Maybe there is a direct greedy algorithm for minimizing maximum lateness?

Greedy Algorithm for Minimizing Lateness

```
Initially $R$ is the set of all requests
$curr\_time = 0$
$curr\_late = 0$
while $R$ is not empty
    choose $i \in R$ with earliest deadline
    $curr\_time = curr\_time + t_i$
    $late = curr\_time - d_i$
    $curr\_late = \max(late, curr\_late)$
return $curr\_late$
```

Exercise: argue directly that above algorithm is correct (see book).

Can be easily implemented in $\mathbf{O(n \log n)}$ time after sorting jobs.

Greedy Analysis: Overview

- Greedy's first step leads to an optimum solution. Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.
- Greedy algorithm stays ahead. Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
- Structural property of solution. Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning.
- Exchange argument. Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.

Takeaway Points

- Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.
- *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.
- Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.