Chapter 8

Dynamic Programming

CS 473: Fundamental Algorithms, Spring 2011
February 15, 2011

8.1 Longest Increasing Subsequence

8.1.1 Longest Increasing Subsequence

8.1.1.1 Sequences

Definition 8.1.1 *Sequence: an ordered list $a_1, a_2, \ldots, a_n$.* **Length** *of a sequence is number of elements in the list.*

Definition 8.1.2 $a_{i_1}, \ldots, a_{i_k}$ *is a* **subsequence** *of* $a_1, \ldots, a_n$ *if* $1 \leq i_1 < \ldots < i_k \leq n$.

Definition 8.1.3 *A sequence is* **increasing** *if* $a_1 < a_2 < \ldots < a_n$. *It is* **non-decreasing** *if* $a_1 \leq a_2 \leq \ldots \leq a_n$. *Similarly* **decreasing** *and non-increasing.*

8.1.2 Sequences

8.1.2.1 Example...

Example 8.1.4 *(A) Sequence: 6, 3, 5, 2, 7, 8, 1*
(B) Subsequence: 5, 2, 1
(C) Increasing sequence: 3, 5, 9
(D) Increasing subsequence: 2, 7, 8

8.1.2.2 Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an *increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

Example 8.1.5 *(A) Sequence: 6, 3, 5, 2, 7, 8, 1*
(B) Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
(C) Longest increasing subsequence: 3, 5, 7, 8

8.1.2.3 Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array A

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.
2^n subsequences of a sequence of length n and $O(n)$ time to check if a given sequence is increasing.

8.1.3 Recursive Approach: Take 1

8.1.3.1 LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?
LIS $(A[1..n])$:
(A) Case 1: does not contain a_n in which case LIS $(A[1..n])$ = LIS $(A[1..(n-1)])$
(B) Case 2: contains a_n in which case LIS $(A[1..n])$ is not so clear.
Observation: if a_n is in the longest increasing subsequence then all the elements before it must be smaller.

8.1.3.2 Recursive Approach: Take 1

```
algLIS(A[1..n]):
    if (n = 0) then return 0
    m = algLIS(A[1..(n − 1)])
    B is subsequence of A[1..(n − 1)] with
            only elements less than a_n
    (* let h be size of B, h ≤ n-1 *)
    m = max(m, 1 + algLIS(B[1..h]))
    Output m
```

Recursion for running time: $T(n) \leq 2T(n-1) + O(n)$.
Easy to see that $T(n)$ is $O(n2^n)$.

8.1.3.3 Recursive Approach: Take 2

LIS($A[1..n]$):
(A) Case 1: does not contain a_n in which case LIS($A[1..n]$) = LIS($A[1..(n-1)]$)
(B) Case 2: contains a_n in which case LIS($A[1..n]$) is not so clear.

Observation 8.1.6 *For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than a_n. This suggests that a more general problem is* **LIS_smaller**($A[1..n], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

8.1.3.4 Recursive Approach: Take 2

LIS_smaller($A[1..n], x$) : length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
```

```
LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
```

Recursion for running time: $T(n) \leq 2T(n-1) + O(1)$.
Question: Is there any advantage?

8.1.3.5 Recursive Algorithm: Take 2

Observation: The number of *different* subproblems generated by **LIS_smaller**($A[1..n], x$) is $O(n^2)$. Memoization the recursive algorithm leads to an $O(n^2)$ running time!

Question: What are the recursive subproblem generated by **LIS_smaller**($A[1..n], x$)?
(A) For $0 \leq i < n$ **LIS_smaller**($A[1..i], y$) where y is either x or one of $A[i+1], \ldots, A[n]$.

Observation: previous recursion also generates only $O(n^2)$ subproblems. Slightly harder to see.

8.1.3.6 Recursive Algorithm: Take 3

LISEnding($A[1..n]$): length of longest increasing sub-sequence that *ends* in $A[n]$.

Question: can we obtain a recursive expression?

$$\textbf{LISEnding}(A[1..n]) = \max_{i:A[i]<A[n]} \left(1 + \textbf{LISEnding}(A[1..i])\right)$$

8.1.3.7 Recursive Algorithm: Take 3

```
LIS_ending(A[1..n]):
    if (n = 0) return 0
    m = 1
    for i = 1 to n - 1 do
        if (A[i] < A[n]) then
            m = max(m, 1 + LIS_ending(A[1..i]))

    return m
```

```
LIS(A[1..n]):
    return max_{i=1}^{n} LIS_ending(A[1...i])
```

Question: How many distinct subproblems generated by **LISEnding**$(A[1..n])$? n.

8.1.3.8 Iterative Algorithm via Memoization

Compute the values **LISEnding**$(A[1..i])$ iteratively in a bottom up fashion.

```
LIS_ending(A[1..n]):
    Array L[1..n]
        (* L[i] = value of LIS_ending(A[1..i]) *)
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i - 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
    return L
```

```
LIS(A[1..n]):
    L = LIS_ending(A[1..n])
    return the maximum value in L
```

8.1.3.9 Iterative Algorithm via Memoization

Simplifying:

```
LIS(A[1..n]):
    Array L[1..n]
            (* L[i] stores the value LIS-Ending(A[1..i]) *)
    m = 0
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i - 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
        m = max(m, L[i])
    return m
```

Correctness: Via induction following the recursion
Running time: $O(n^2)$, *Space:* $\Theta(n)$

8.1.3.10 Example

Example 8.1.7 *(A) Sequence: 6, 3, 5, 2, 7, 8, 1*

(B) *Longest increasing subsequence: 3, 5, 7, 8*

(A) $L[i]$ is value of longest increasing subsequence ending in $A[i]$
(B) Recursive algorithm computes $L[i]$ from $L[1]$ to $L[i-1]$
(C) Iterative algorithm builds up the values from $L[1]$ to $L[n]$

8.1.3.11 Memoizing LIS_smaller

```
LIS(A[1..n]):
    A[n+1] = ∞ (* add a sentinel at the end *)
    Array L[(n+1),(n+1)] (* two-dimensional array*)
        (* L[i,j] for j ≥ i stores the value LIS_smaller(A[1..i],A[j]) *)
    for j = 1 to n+1 do
        L[0,j] = 0
    for i = 1 to n+1 do
        for j = i to n+1 do
            L[i,j] = L[i-1,j]
            if (A[i] < A[j]) then
                L[i,j] = max(L[i,j], 1 + L[i-1,i])

    return L[n,(n+1)]
```

Correctness: Via induction following the recursion (take 2)
Running time: $O(n^2)$, *Space:* $\Theta(n^2)$

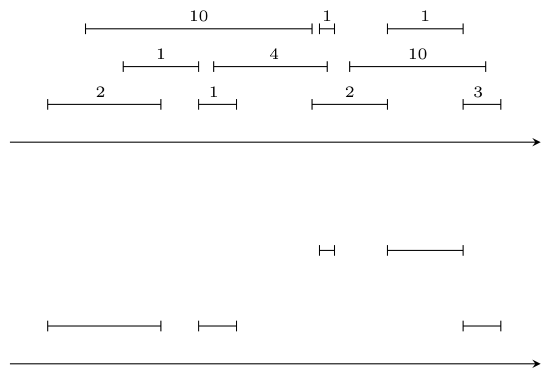
8.1.4 Longest increasing subsequence

8.1.4.1 Another way to get quadratic time algorithm

(A) $G = (\{s, 1, \ldots, n\}, \{\})$: directed graph.
  (A) $\forall i, j$: If $i < j$ and $A[i] < A[j]$ then
      add the edge $i \to j$ to G.
  (B) $\forall i$: Add $s \to i$.
(B) The graph G is a DAG. LIS corresponds to longest path in G starting at s.
(C) We know how to compute this in $O(|V(G)| + |E(G)|) = O(n^2)$.
  Comment: One can compute LIS in $O(n \log n)$ time with a bit more work.

8.1.4.2 Dynamic Programming

(A) Find a "smart" recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
(B) Eliminate recursion and find an iterative algorithm to compute the problems bottom up by storing the intermediate values in an appropriate data structure; need to find the right way or order the subproblem evaluation.



(C) Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value. Evaluate the total running time.
(D) Optimize the resulting algorithm further

8.2 Weighted Interval Scheduling

8.2.1 Weighted Interval Scheduling

8.2.2 The Problem

8.2.2.1 Weighted Interval Scheduling

Input A set of jobs with start times, finish times and *weights* (or profits).

Goal Schedule jobs so that total weight of jobs is maximized.

(A) Two jobs with overlapping intervals cannot both be scheduled!

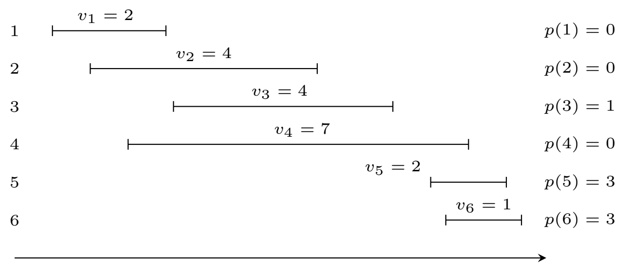
8.2.3 Greedy Solution

8.2.3.1 Interval Scheduling

Input A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight 1.

Goal Schedule as many jobs as possible.

(A) Greedy strategy of considering jobs according to finish times produces optimal schedule (to be seen later).

Example 8.2.2 

8.2.3.2 Greedy Strategies

(A) Earliest finish time first
(B) Largest weight/profit first
(C) Largest weight to length ratio first
(D) Shortest length first
(E) ...

None of the above strategies lead to an optimum solution.

Moral: Greedy strategies often don't work!

8.2.3.3 Reduction to Max Weight Independent Set Problem

(A) Given weighted interval scheduling instance I create an instance of max weight independent set on a graph $G(I)$ as follows.
  (A) For each interval i create a vertex v_i with weight w_i.
  (B) Add an edge between v_i and v_j if i and j overlap.
(B) *Claim:* max weight independent set in $G(I)$ has weight equal to max weight set of intervals in I that do not overlap

We do not know an efficient (polynomial time) algorithm for independent set! Can we take advantage of the interval structure to find an efficient algorithm?

8.2.4 Recursive Solution

8.2.4.1 Conventions

Definition 8.2.1 *(A) Let the requests be sorted according to finish time, i.e., $i < j$ implies $f_i \leq f_j$*
(B) Define $p(j)$ to be the largest i (less than j) such that job i and job j are not in conflict

8.2.4.2 Towards a Recursive Solution

Observation 8.2.3 *Consider an optimal schedule $\mathcal{O}$*

Case $n \in \mathcal{O}$: *None of the jobs between n and $p(n)$ can be scheduled. Moreover $\mathcal{O}$ must contain an optimal schedule for the first $p(n)$ jobs.*

Case $n \notin \mathcal{O}$: *$\mathcal{O}$ is an optimal schedule for the first $n - 1$ jobs.*

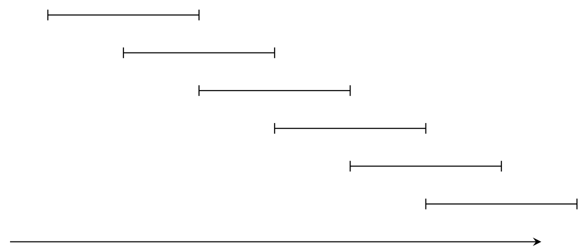


Figure 8.1: Bad instance for recursive algorithm

8.2.4.3 A Recursive Algorithm

Let O_i be value of an optimal schedule for the first i jobs.

```
Schedule(n):
    if n = 0 then return 0
    if n = 1 then return w(v_1)
    O_p(n) ← Schedule(p(n))
    O_{n-1} ← Schedule(n - 1)
    if (O_p(n) + w(v_n) < O_{n-1}) then
        O_n = O_{n-1}
    else
        O_n = O_p(n) + w(v_n)
    return O_n
```

Time Analysis
Running time is $T(n) = T(p(n)) + T(n-1) + O(1)$ which is ...

8.2.4.4 Bad Example

Running time on this instance is

$$T(n) = T(n-1) + T(n-2) + O(1) = \Theta(\phi^n)$$

where $\phi \approx 1.618$ is the golden ratio.
(Because... $T(n)$ is the n Fibonacci number.)

8.2.4.5 Analysis of the Problem

8.2.5 Dynamic Programming

8.2.5.1 Memo(r)ization

Observation 8.2.4 *(A) Number of different sub-problems in recursive algorithm is $O(n)$; they are $O_1, O_2, \ldots, O_{n-1}$*
(B) Exponential time is due to recomputation of solutions to sub-problems

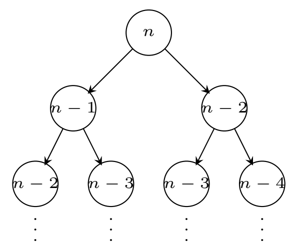


Figure 8.2: Label of node indicates size of sub-problem. Tree of sub-problems grows very quickly

Solution
Store optimal solution to different sub-problems, and perform recursive call *only* if not already computed.

8.2.5.2 Recursive Solution with Memoization

```
schdIMem(j)
    if j = 0 then return 0
    if M[j] is defined then (* sub-problem already solved *)
        return M[j]
    if M[j] is not defined then
        M[j] = max(w(v_j) + schdIMem(p(j)),   schdIMem(j - 1))
        return M[j]
```

Time Analysis

- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one $M[\cdot]$
- Initially no entry of $M[]$ is filled; at the end all entries of `M[]` are filled
- So total time is $O(n)$

8.2.5.3 Automatic Memoization

Fact
Many functional languages (like LISP) automatically do memoization for recursive function calls!

8.2.5.4 Back to Weighted Interval Scheduling

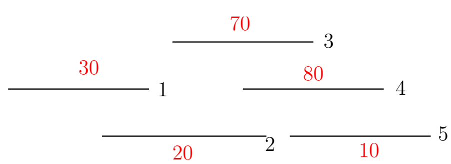
Iterative Solution

$M[0] = 0$
for $i = 1$ `to` n **do**
    $M[i] = \max\Big(w(v_i) + M[p(i)], M[i-1]\Big)$

M: table of subproblems

(A) Implicitly dynamic programming fills the values of M
(B) Recursion determines order in which table is filled up
(C) Think of decomposing problem first (recursion) and then worry about setting up table — this comes naturally from recursion

8.2.5.5 Example



$p(5) = 2$, $p(4) = 1$, $p(3) = 1$, $p(2) = 0$, $p(1) = 0$

8.2.6 Computing Solutions

8.2.6.1 Computing Solutions + First Attempt

(A) Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

```
M[0] = 0
S[0] is empty schedule
for i = 1 to n do
    M[i] = max( w(v_i) + M[p(i)], M[i − 1] )
    if w(v_i) + M[p(i)] < M[i − 1] then
        S[i] = S[i − 1]
    else
        S[i] = S[p(i)] ∪ {i}
```

(B) Naïvely updating $S[]$ takes $O(n)$ time
(C) Total running time is $O(n^2)$
(D) Using pointers running time can be improved to $O(n)$.

8.2.6.2 Computing Implicit Solutions

Observation 8.2.5 *Solution can be obtained from $M[]$ in $O(n)$ time, without any additional information*

```
findSolution( j )
    if (j = 0) then return empty schedule
    if (v_j + M[p(j)] > M[j − 1]) then
        return findSolution(p(j)) ∪ {j}
    else
        return findSolution(j − 1)
```

Makes $O(n)$ recursive calls, so **findSolution** *runs in $O(n)$ time.*

8.2.6.3 Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

(A) keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
(B) once the optimum values are computed, go back and use the decision values to compute an optimum solution.

Question: What is the decision in computing $M[i]$?

A: Whether to include i or not.