# Dynamic Programming

Lecture 8

February 15, 2011

Part I

# Longest Increasing Subsequence

# Sequences

## Definition

Sequence: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

## Definition

$a_{i_1}, \ldots, a_{i_k}$ is a **subsequence** of $a_1, \ldots, a_n$ if $1 \leq i_1 < \ldots < i_k \leq n$.

## Definition

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and non-increasing.

# Sequences

## Example...

### Example

- Sequence: 6, 3, 5, 2, 7, 8, 1
- Subsequence: 5, 2, 1
- Increasing sequence: 3, 5, 9
- Increasing subsequence: 2, 7, 8

# Longest Increasing Subsequence Problem

**Input** A sequence of numbers $a_1, a_2, \ldots, a_n$

**Goal** Find an *increasing subsequence* $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

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- Sequence: 6, 3, 5, 2, 7, 8, 1
- Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
- Longest increasing subsequence: 3, 5, 7, 8

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# Naïve Enumeration

Assume $a_1, a_2, \ldots, a_n$ is contained in an array $A$

```
algLISNaive(A[1..n]):
    max = 0
    for each subsequence B of A do
        if B is increasing and |B| > max then
            max = |B|

    Output max
```

Running time: $O(n2^n)$.

$2^n$ subsequences of a sequence of length $n$ and $O(n)$ time to check if a given sequence is increasing.

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# Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS (**A[1..n]**):

- Case 1: does not contain $a_n$ in which case LIS (**A[1..n]**) = LIS (**A[1..(n − 1)]**)
- Case 2: contains $a_n$ in which case LIS (**A[1..n]**) is not so clear.

**Observation:** if $a_n$ is in the longest increasing subsequence then all the elements before it must be smaller.

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    (* let h be size of B, h ≤ n-1 *)
    m = max(m, 1 + algLIS(B[1..h]))
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Recursion for running time: $T(n) \leq 2T(n-1) + O(n)$.

Easy to see that $T(n)$ is $O(n2^n)$.

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## Observation

*For second case we want to find a subsequence in* **A[1..(n − 1)]** *that is restricted to numbers less than* $\mathbf{a_n}$*. This suggests that a more general problem is* **LIS_smaller(A[1..n], x)** *which gives the longest increasing subsequence in* **A** *where each number in the sequence is less than* **x**.

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**LIS_smaller(A[1..n], x) :** length of longest increasing subsequence in **A[1..n]** with all numbers in subsequence less than **x**

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**Observation:** The number of *different* subproblems generated by **LIS_smaller(A[1..n], x)** is $O(n^2)$. Memoization the recursive algorithm leads to an $O(n^2)$ running time!

**Question:** What are the recursive subproblem generated by **LIS_smaller(A[1..n], x)**?

- For $0 \leq i < n$ **LIS_smaller(A[1..i], y)** where **y** is either **x** or one of $A[i+1], \ldots, A[n]$.

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**LISEnding(A[1..n])**: length of longest increasing sub-sequence that *ends* in **A[n]**.

**Question:** can we obtain a recursive expression?

$$\text{LISEnding}(A[1..n]) = \max_{i:A[i]<A[n]} (1 + \text{LISEnding}(A[1..i]))$$

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# Recursive Algorithm: Take 3

```
LIS_ending(A[1..n]):
    if (n = 0) return 0
    m = 1
    for i = 1 to n − 1 do
        if (A[i] < A[n]) then
            m = max(m, 1 + LIS_ending(A[1..i]))

    return m
```

```
LIS(A[1..n]):
    return max_{i=1}^{n} LIS_ending(A[1 . . . i])
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**Question:** How many distinct subproblems generated by LISEnding(A[1..n])? n.

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# Iterative Algorithm via Memoization

Compute the values **LISEnding(A[1..i])** iteratively in a bottom up fashion.

```
LIS_ending(A[1..n]):
    Array L[1..n]
        (* L[i] = value of LIS_ending(A[1..i]) *)
    for i = 1 to n do
        L[i] = 1
        for j = 1 to i − 1 do
            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
    return L
```

```
LIS(A[1..n]):
    L = LIS_ending(A[1..n])
    return the maximum value in L
```

# Iterative Algorithm via Memoization

Simplifying:

```
LIS(A[1..n]):
    Array L[1..n]
            (* L[i] stores the value LIS-Ending(A[1..i]) *)
    m = 0
    for i = 1 to n do
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            if (A[j] < A[i]) do
                L[i] = max(L[i], 1 + L[j])
        m = max(m, L[i])
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Correctness: Via induction following the recursion

Running time: $O(n^2)$, Space: $\Theta(n)$

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    for j = 1 to n+1 do
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# Longest increasing subsequence

## Another way to get quadratic time algorithm

- $G = (\{s, 1, \ldots, n\}, \{\})$: directed graph.
  - $\forall i, j$: If $i < j$ and $A[i] < A[j]$ then add the edge $i \to j$ to $G$.
  - $\forall i$: Add $s \to i$.
- The graph $G$ is a DAG. LIS corresponds to longest path in $G$ starting at $s$.
- We know how to compute this in $O(|V(G)| + |E(G)|) = O(n^2)$.

Comment: One can compute LIS in $O(n \log n)$ time with a bit more work.

# Dynamic Programming

- Find a "smart" recursion for the problem in which the number of distinct subproblems is small; polynomial in the original problem size.
- Eliminate recursion and find an iterative algorithm to compute the problems bottom up by storing the intermediate values in an appropriate data structure; need to find the right way or order the subproblem evaluation.
- Estimate the number of subproblems, the time to evaluate each subproblem and the space needed to store the value. Evaluate the total running time.
- Optimize the resulting algorithm further

Part II

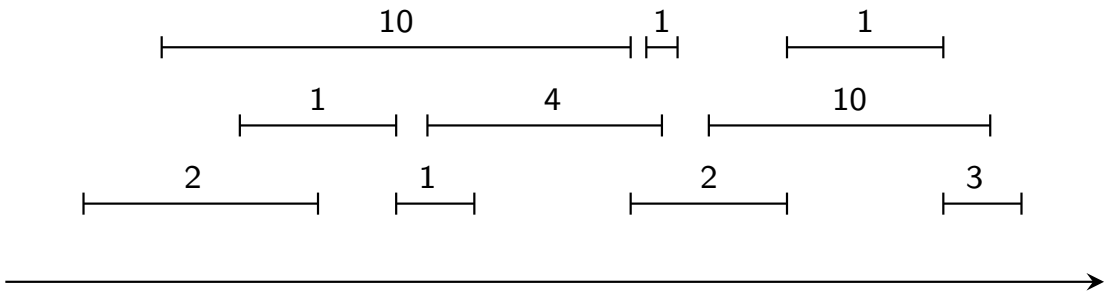
# Weighted Interval Scheduling

# Weighted Interval Scheduling

**Input** A set of jobs with start times, finish times and *weights* (or profits).

**Goal** Schedule jobs so that total weight of jobs is maximized.

- Two jobs with overlapping intervals cannot both be scheduled!

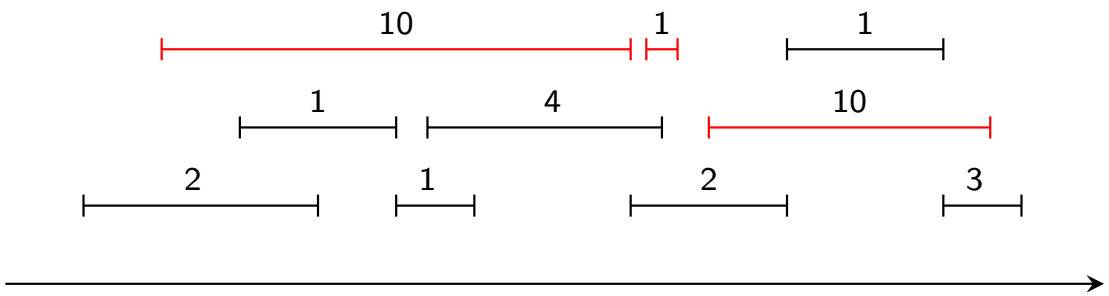


# Weighted Interval Scheduling

**Input** A set of jobs with start times, finish times and *weights* (or profits).

**Goal** Schedule jobs so that total weight of jobs is maximized.

- Two jobs with overlapping intervals cannot both be scheduled!

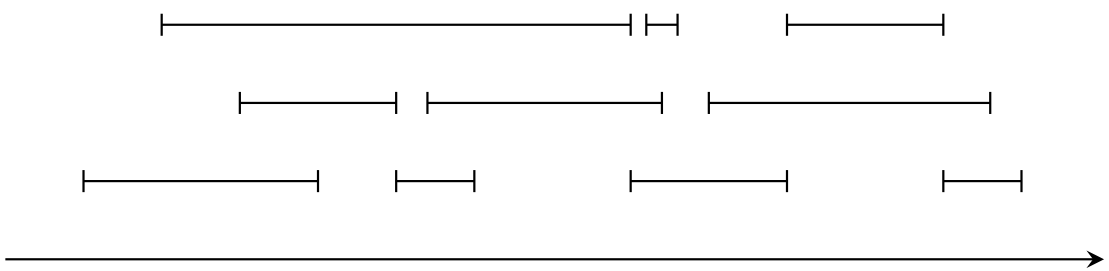


# Interval Scheduling

**Input** A set of jobs with start and finish times to be scheduled on a resource; special case where all jobs have weight **1**.

**Goal** Schedule as many jobs as possible.

- Greedy strategy of considering jobs according to finish times produces optimal schedule (to be seen later).



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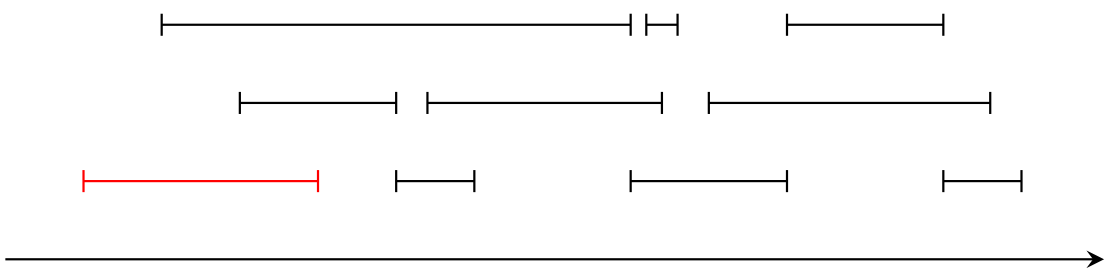

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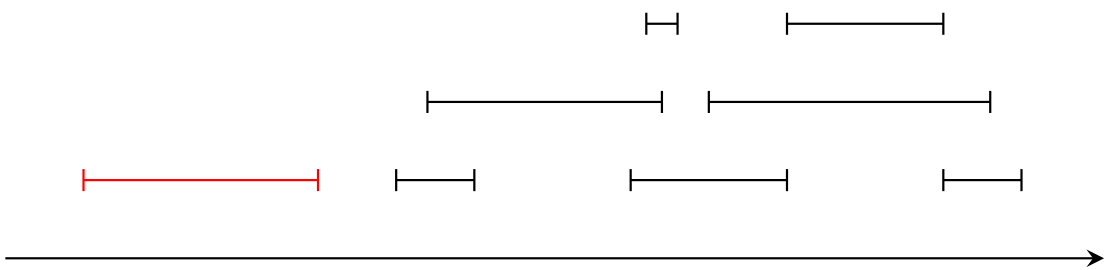

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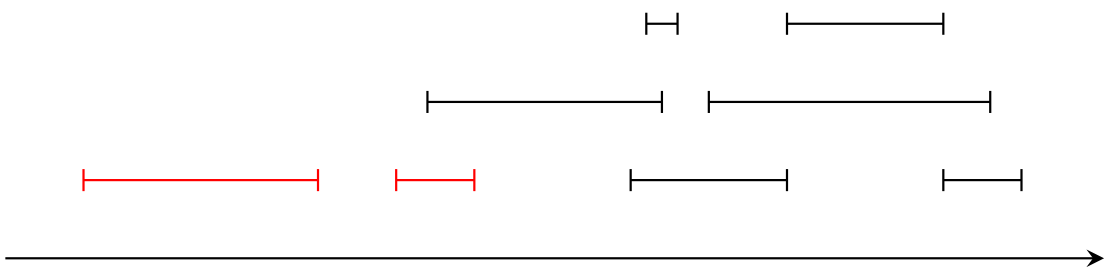

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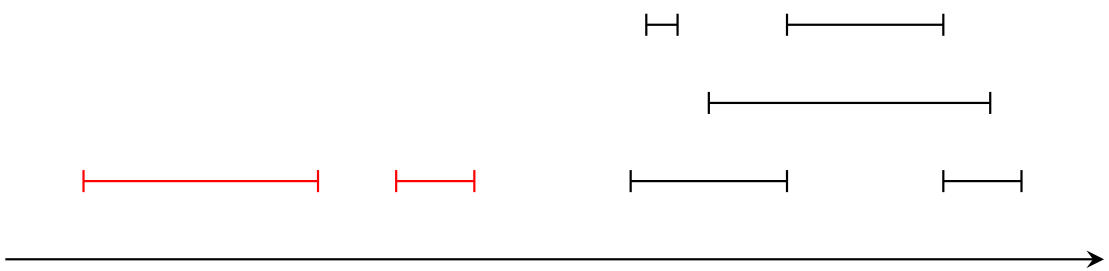

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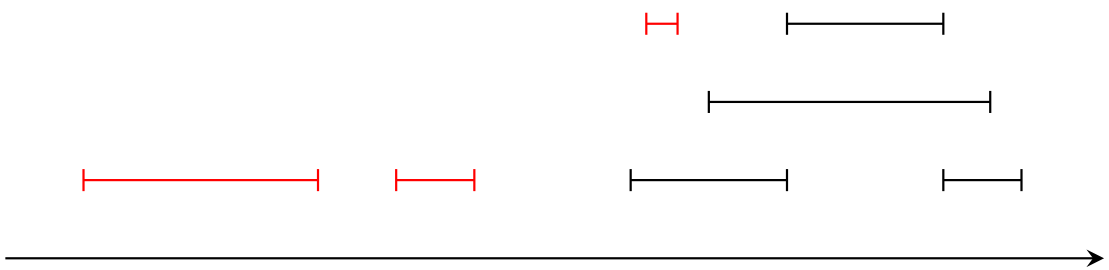

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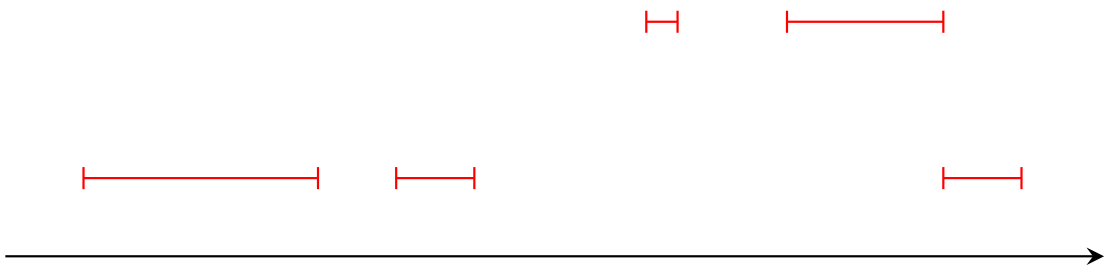

# Greedy Strategies

- Earliest finish time first
- Largest weight/profit first
- Largest weight to length ratio first
- Shortest length first
- ...

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# Reduction to Max Weight Independent Set Problem

- Given weighted interval scheduling instance $I$ create an instance of max weight independent set on a graph $G(I)$ as follows.
  - For each interval $i$ create a vertex $v_i$ with weight $w_i$.
  - Add an edge between $v_i$ and $v_j$ if $i$ and $j$ overlap.
- Claim: max weight independent set in $G(I)$ has weight equal to max weight set of intervals in $I$ that do not overlap

We do not know an efficient (polynomial time) algorithm for independent set! Can we take advantage of the interval structure to find an efficient algorithm?

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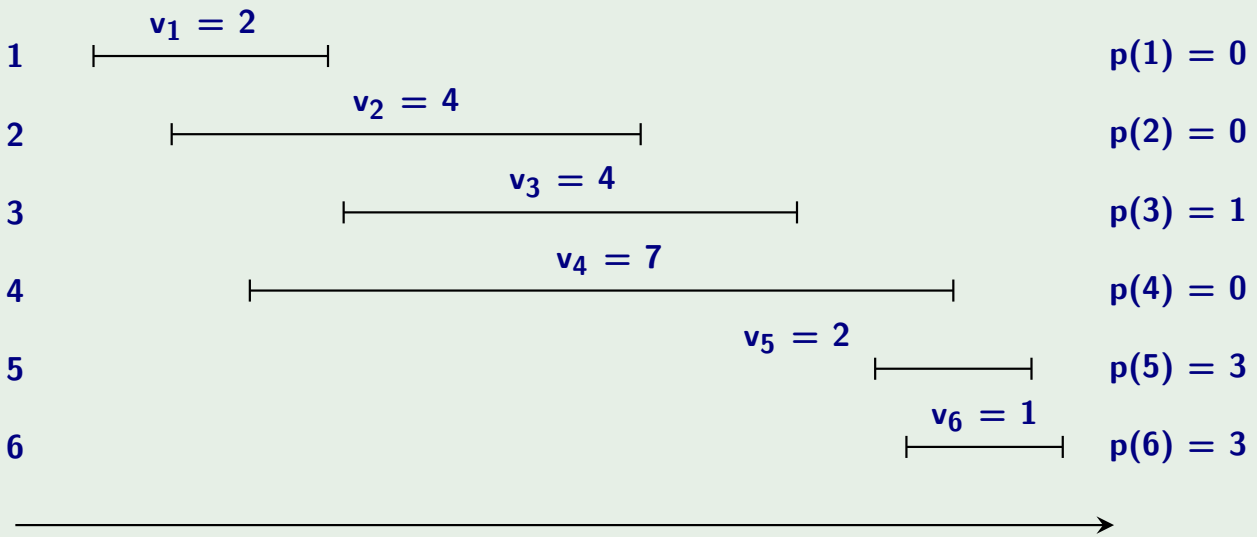
# Conventions

## Definition

- Let the requests be sorted according to finish time, i.e., $i < j$ implies $f_i \leq f_j$
- Define $p(j)$ to be the largest $i$ (less than $j$) such that job $i$ and job $j$ are not in conflict

## Example



# Towards a Recursive Solution

## Observation

*Consider an optimal schedule* $\mathcal{O}$

Case $\mathbf{n} \in \mathcal{O}$ : *None of the jobs between* $\mathbf{n}$ *and* $\mathbf{p(n)}$ *can be scheduled. Moreover* $\mathcal{O}$ *must contain an optimal schedule for the first* $\mathbf{p(n)}$ *jobs.*

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# A Recursive Algorithm

Let $O_i$ be value of an optimal schedule for the first $i$ jobs.

```
Schedule(n):
    if n = 0 then return 0
    if n = 1 then return w(v_1)
    O_{p(n)} ← Schedule(p(n))
    O_{n-1} ← Schedule(n - 1)
    if (O_{p(n)} + w(v_n) < O_{n-1}) then
        O_n = O_{n-1}
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    return O_n
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## Time Analysis

Running time is $T(n) = T(p(n)) + T(n-1) + O(1)$ which is ...

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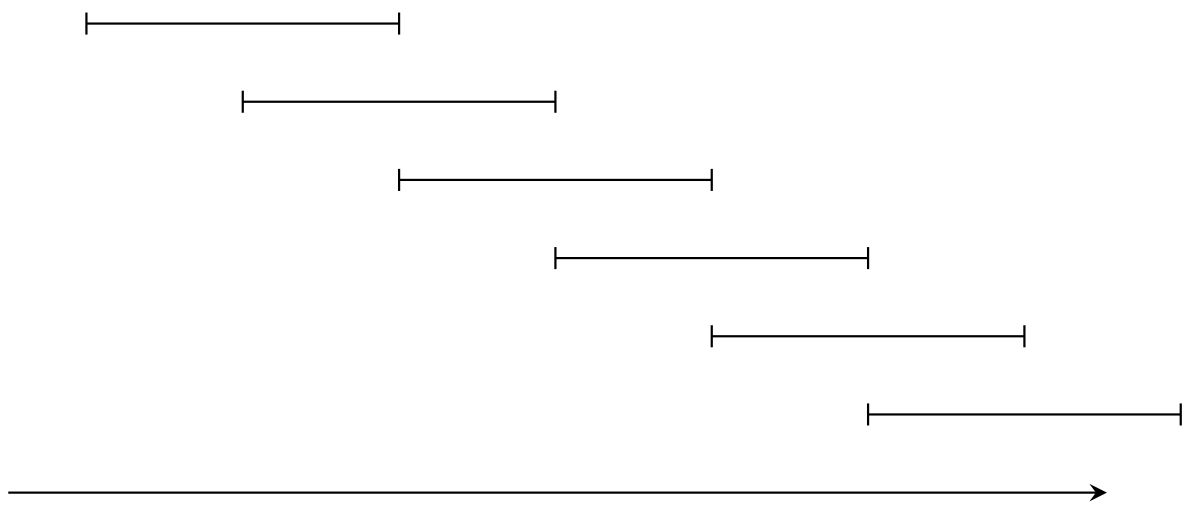

Figure: Bad instance for recursive algorithm

Running time on this instance is

$$\mathbf{T(n) = T(n-1) + T(n-2) + O(1)} = \Theta(\phi^n)$$

where $\phi \approx \mathbf{1.618}$ is the golden ratio.

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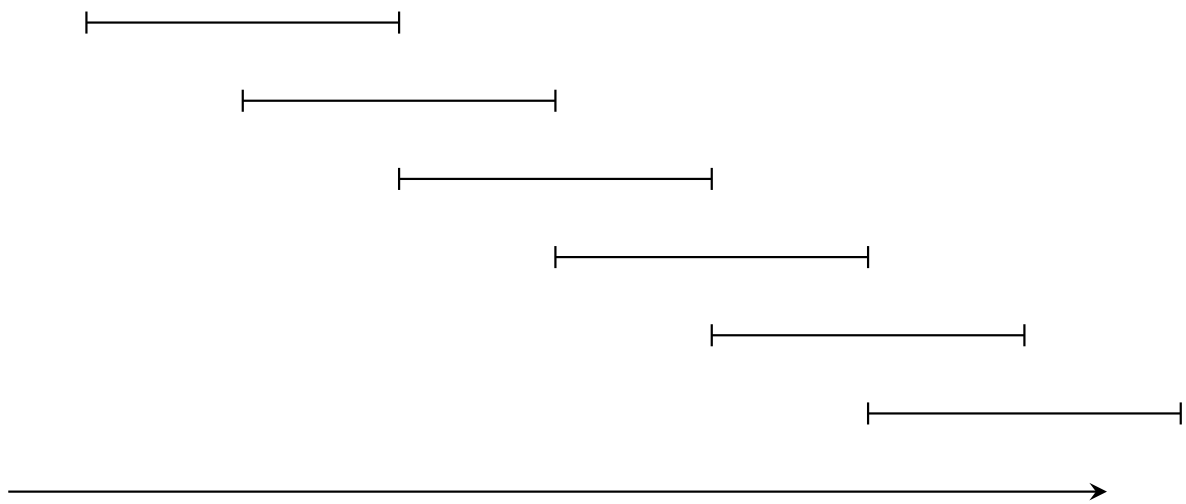

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# Analysis of the Problem

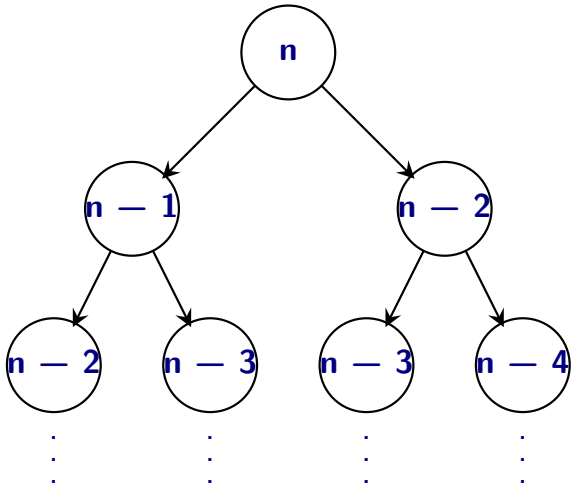


Figure: Label of node indicates size of sub-problem. Tree of sub-problems grows very quickly

# Memo(r)ization

## Observation

- *Number of different sub-problems in recursive algorithm is* $\mathbf{O(n)}$*; they are* $\mathbf{O_1, O_2, \ldots, O_{n-1}}$
- *Exponential time is due to recomputation of solutions to sub-problems*

## Solution

Store optimal solution to different sub-problems, and perform recursive call only if not already computed.

# Recursive Solution with Memoization

```
schdIMem(j)
    if j = 0 then return 0
    if M[j] is defined then (* sub-problem already solved *)
        return M[j]
    if M[j] is not defined then
        M[j] = max(w(v_j) + schdIMem(p(j)), schdIMem(j − 1))
        return M[j]
```

## Time Analysis

- Each invocation, $O(1)$ time plus: either return a computed value, or generate 2 recursive calls and fill one $M[\cdot]$
- Initially no entry of $M[]$ is filled, at the end all $n$ entries of $M[]$ are filled
- So total time is $O(n)$

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# Automatic Memoization

## Fact

Many functional languages (like LISP) automatically do memoization for recursive function calls!

# Back to Weighted Interval Scheduling

## Iterative Solution

$M[0] = 0$
**for** $i = 1$ `to` $n$ **do**
$\quad M[i] = \max\left(w(v_i) + M[p(i)], M[i-1]\right)$

M: table of subproblems

- Implicitly dynamic programming fills the values of M
- Recursion determines order in which table is filled up
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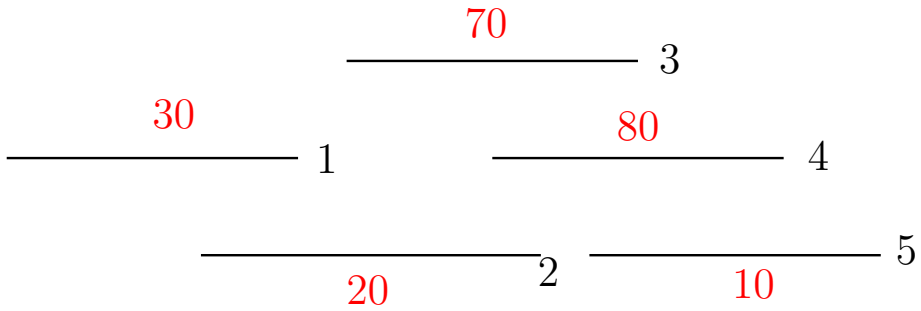


$p(5) = 2$, $p(4) = 1$, $p(3) = 1$, $p(2) = 0$, $p(1) = 0$

# Computing Solutions + First Attempt

- Memoization + Recursion/Iteration allows one to compute the optimal value. What about the actual schedule?

```
M[0] = 0
S[0] is empty schedule
for i = 1 to n do
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- Naïvely updating $S[]$ takes $O(n)$ time
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# Computing Implicit Solutions

## Observation

*Solution can be obtained from* **M[]** *in* **O(n)** *time, without any additional information*

```
findSolution( j )
    if (j = 0) then return empty schedule
    if (v_j + M[p(j)] > M[j − 1]) then
        return findSolution(p(j)) ∪ {j}
    else
        return findSolution(j − 1)
```

*Makes* **O(n)** *recursive calls, so* **findSolution** *runs in* **O(n)** *time.*

# Computing Implicit Solutions

A generic strategy for computing solutions in dynamic programming:

- keep track of the *decision* in computing the optimum value of a sub-problem. decision space depends on recursion
- once the optimum values are computed, go back and use the decision values to compute an optimum solution.

Question: What is the decision in computing **M[i]**?

A: Whether to include **i** or not.

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        Decision[i] = 1 (* 1: i included in solution M[i] *)
    else
        Decision[i] = 0 (* 0: i not included in solution M[i] *)

S = ∅, i = n
while (i > 0) do
    if (Decision[i] = 1) then
        S = S ∪ {i}
        i = p(i)
    else
        i = i − 1
return S
```

# Notes

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