Chapter 7

Binary Search, Introduction to Dynamic Programming

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7.1 Exponentiation, Binary Search

7.2 Exponentiation

7.2.0.1 Exponentiation

Input  Two numbers: \(a\) and integer \(n \geq 0\)

Goal  Compute \(a^n\)

Obvious algorithm:

\[
\text{SlowPow}(a, n):
\]
\[
\begin{align*}
x & = 1; \\
\text{for } i = 1 \text{ to } n \text{ do} \\
& \quad x = x \times a \\
\text{Output } x
\end{align*}
\]

\(O(n)\) multiplications.

7.2.0.2 Fast Exponentiation

Observation: \(a^n = a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil} = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor + a^{\lfloor n/2 \rfloor - \lfloor n/2 \rfloor}}.\)
FastPow(a,n):
   if (n = 0) return 1
   x = FastPow(a,\lfloor n/2 \rfloor)
   x = x*x
   if (n is odd) then
      x = x * a
   return x

$T(n)$: number of multiplications for $n$

$$T(n) \leq T(\lfloor n/2 \rfloor) + 2$$

$T(n) = \Theta(\log n)$.

### 7.2.0.3 Complexity of Exponentiation

**Question:** Is SlowPow() a polynomial time algorithm? FastPow?

Input size: $\log a + \log n$
Output size: $n \log a$. Not necessarily polynomial in input size!

Both FastPow and FastPow are polynomial in output size.

### 7.2.0.4 Exponentiation modulo a given number

Exponentiation in applications:

**Input** Three integers: $a$, $n \geq 0$, $p \geq 2$ (typically a prime)

**Goal** Compute $a^n \mod p$

Input size: $\Theta(\log a + \log n + \log p)$
Output size: $O(\log p)$ and hence polynomial in input size.

**Observation:** $xy \mod p = ((x \mod p)(y \mod p)) \mod p$

### 7.2.0.5 Exponentiation modulo a given number

**Input** Three integers: $a$, $n \geq 0$, $p \geq 2$ (typically a prime)

**Goal** Compute $a^n \mod p$
FastPowMod is a polynomial time algorithm. SlowPowMod is not (why?).

7.3 Binary Search

7.3.0.6 Binary Search in Sorted Arrays

Input Sorted array $A$ of $n$ numbers and number $x$

Goal Is $x$ in $A$?

```
BinarySearch(A[a..b], x):
    if (b-a <= 0) return NO
    mid = A[⌊(a+b)/2⌋]
    if (x = mid) return YES
    else if (x < mid) return BinarySearch(A[a..⌊(a+b)/2⌋−1],x)
    else return BinarySearch(A[⌊(a+b)/2⌋+1..b],x)
```

Analysis: $T(n) = T(⌊n/2⌋) + O(1)$. $T(n) = O(\log n)$.

Observation: After $k$ steps, size of array left is $n/2^k$

7.3.0.7 Another common use of binary search

- **Optimization version:** find solution of best (say minimum) value
- **Decision version:** is there a solution of value at most a given value $v$?

Reduce optimization to decision (may be easier to think about):

- Given instance $I$ compute upper bound $U(I)$ on best value
- Compute lower bound $L(I)$ on best value
- Do binary search on interval $[L(I), U(I)]$ using decision version as black box
- $O(\log(U(I) - L(I)))$ calls to decision version if $U(I), L(I)$ are integers
7.3.0.8 Example

- **Problem:** shortest paths in a graph.
- **Decision version:** given $G$ with non-negative integer edge lengths, nodes $s, t$ and bound $B$, is there an $s$-$t$ path in $G$ of length at most $B$?
- **Optimization version:** find the length of a shortest path between $s$ and $t$ in $G$.

*Question:* given a black box algorithm for the decision version, can we obtain an algorithm for the optimization version?

7.3.0.9 Example continued

*Question:* given a black box algorithm for the decision version, can we obtain an algorithm for the optimization version?

- Let $U$ be maximum edge length in $G$.
- Minimum edge length is $L$.
- $s$-$t$ shortest path length is at most $(n - 1)U$ and at least $L$.
- Apply binary search on the interval $[L, (n - 1)U]$ via the algorithm for the decision problem.
- $O(\log((n - 1)U - L))$ calls to the decision problem algorithm sufficient. Polynomial in input size.

7.4 Introduction to Dynamic Programming

7.4.0.10 Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

- reduce problem to a *smaller* instance of itself
- self-reduction
- Problem instance of size $n$ is reduced to one or more instances of size $n - 1$ or less.
- For termination, problem instances of small size are solved by some other method as *base cases*
7.4.0.11 Recursion in Algorithm Design

- **Tail Recursion**: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms. Examples: Interval scheduling, MST algorithms, etc.

- **Divide and Conquer**: problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem.

- **Dynamic Programming**: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use memoization to avoid recomputation of common solutions leading to iterative bottom-up algorithm.

7.5 Fibonacci Numbers

7.5.0.12 Fibonacci Numbers

Fibonacci numbers defined by recurrence:

\[ F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1. \]

These numbers have many interesting and amazing properties.

A journal *The Fibonacci Quarterly*!

- \[ F(n) = (\phi^n - (1 - \phi)^n)/\sqrt{5} \text{ where } \phi \text{ is the golden ratio } (1 + \sqrt{5})/2 \simeq 1.618. \]

- \[ \lim_{n \to \infty} F(n+1)/F(n) = \phi \]

*Question*: Given \( n \), compute \( F(n) \).

7.5.0.13 Recursive Algorithm for Fibonacci Numbers

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n-1) + Fib(n-2)
```

Running time? Let \( T(n) \) be the number of additions in \( \text{Fib}(n) \).

\[ T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0 \]
Roughly same as $F(n)$  

$$T(n) = \Theta(\phi^n)$$  

The number of additions is exponential in $n$. Can we do better?

### 7.5.0.14 An iterative algorithm for Fibonacci numbers

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        F[0] = 0
        F[1] = 1
        for i = 2 to n do
            F[i] = F[i-1] + F[i-2]
        return F[n]
```

What is the running time of the algorithm? $O(n)$ additions.

### 7.5.0.15 What is the difference?

- Recursive algorithm is computing the same numbers again and again.
- Iterative algorithm is storing computed values and building bottom up the final value. *Memoization.*

Dynamic Programming: finding a recursion that can be *effectively/efficiently* memoized

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.

### 7.5.0.16 Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (Fib(n) was previously computed)
        return stored value of Fib(n)
    else
        return Fib(n-1) + Fib(n-2)
```
How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)

7.5.0.17 Automatic explicit memoization

Initialize table/array \( M \) of size \( n \) such that \( M[i] = -1 \) for \( 0 \leq i < n \)

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (M[n] \neq -1) (* M[n] has stored value of Fib(n) *)
        return M[n]
    else
        M[n] = Fib(n-1) + Fib(n-2)
        return M[n]
```

Need to know upfront the number of subproblems to allocate memory

7.5.0.18 Automatic implicit memoization

Initialize a (dynamic) dictionary data structure \( D \) to empty

```python
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else if (n is already in D)
        return value stored with n in D
    else
        val = Fib(n-1) + Fib(n-2)
        Store (n, val) in D
        return val
```

7.5.0.19 Explicit vs Implicit Memoization

- Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.
- Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system
  - need to pay overhead of datastructure
  - Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.
7.5.0.20 Back to Fibonacci Numbers

Is the iterative algorithm a polynomial time algorithm? Does it take $O(n)$ time?

- input is $n$ and hence input size is $\Theta(\log n)$
- output is $F(n)$ and output size is $\Theta(n)$. Why?
- Hence output size is exponential in input size so no polynomial time algorithm possible!
- Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?
- Running time of recursive algorithm is $O(n\phi^n)$ but can in fact shown to be $O(\phi^n)$ by being careful. Doubly exponential in input size and exponential even in output size.

7.6 Brute Force Search, Recursion and Backtracking

7.6.0.21 Maximum Independent Set in a Graph

**Definition 7.6.1** Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in $S$. That is, if $u, v \in S$ then $(u, v) \notin E$.

Some independent sets in graph above:

7.6.0.22 Maximum Independent Set Problem

**Input** Graph $G = (V, E)$

**Goal** Find maximum sized independent set in $G$
7.6.0.23 Maximum Weight Independent Set Problem

**Input** Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

**Goal** Find maximum weight independent set in $G$

![Graph](image)

7.6.0.24 Maximum Weight Independent Set Problem

- No one knows an *efficient* (polynomial time) algorithm for this problem
- Problem is **NP-Complete** and it is *believed* that there is no polynomial time algorithm

A *brute-force* algorithm: try all subsets of vertices.

7.6.0.25 Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

```plaintext
MaxIndSet(G = (V, E)):
  max = 0
  for each subset $S \subseteq V$ do
    check if $S$ is an independent set
    if $S$ is an independent set and $w(S) > max$ then
      $max = w(S)$
  Output $max$
```

Running time: suppose $G$ has $n$ vertices and $m$ edges

- $2^n$ subsets of $V$
- checking each subset $S$ takes $O(m)$ time
- total time is $O(m2^n)$
7.6.0.26 A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$.
For a vertex $u$ let $N(u)$ be its neighbours.

Observation 7.6.2 $v_n$: Vertex in the graph.
One of the following two cases is true

Case 1 $v_n$ is in some maximum independent set.

Case 2 $v_n$ is in no maximum independent set.

RecursiveMIS($G$):
if $G$ is empty then Output 0
$a = \text{RecursiveMIS}(G - v_n)$
$b = w(v_n) + \text{RecursiveMIS}(G - v_n - N(v_n))$
Output max($a, b$)

7.6.1 Recursive Algorithms

7.6.1.1 for Maximum Independent Set

Running time:

$$T(n) = T(n - 1) + T\left(n - 1 - \deg(v_n)\right) + O(1 + \deg(v_n))$$

where $\deg(v_n)$ is the degree of $v_n$. $T(0) = T(1) = 1$ is base case.
Worst case is when $\deg(v_n) = 0$ when the recurrence becomes

$$T(n) = 2T(n - 1) + O(1)$$

Solution to this is $T(n) = O(2^n)$.

7.6.1.2 Backtrack Search via Recursion

- Recursive algorithm generates a tree of computation where each node is a smaller problem (subproblem)
- Simple recursive algorithm computes/explores the whole tree blindly in some order.
- Backtrack search is a way to explore the tree intelligently to prune the search space
  - Some subproblems may be so simple that we can stop the recursive algorithm and solve it directly by some other method