Reductions, Recursion and Divide and Conquer

Lecture 5

February 1, 2011

Part I

Reductions and Recursion

Reduction

Reducing problem **A** to problem **B**:

- Algorithm for **A** uses algorithm for **B** as a *black box*

Distinct Elements Problem

Problem Given an array $\mathbf{A}$ of $\mathbf{n}$ integers, are there any *duplicates* in $\mathbf{A}$?

Naive algorithm:

```
for i = 1 to n - 1 do
    for j = i + 1 to n do
        if (A[i] = A[j])
            return YES
return NO
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Running time: $O(n^2)$

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    if ($\mathbf{A[i] = A[j]}$)
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return NO

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Reduction to Sorting

```
Sort A
for i = 1 to n − 1 do
    if (A[i] = A[i + 1]) then
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return NO
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Running time: $O(n)$ plus time to sort an array of n numbers

Important point: algorithm uses sorting as a black box

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Two sides of Reductions

Suppose problem **A** reduces to problem **B**

- Positive direction: Algorithm for **B** implies an algorithm for **A**
- Negative direction: Suppose there is no "efficient" algorithm for **A** then it implies no efficient algorithm for **B** (technical condition for reduction time necessary for this)

Example: Distinct Elements reduces to Sorting in $O(n)$ time

- An $O(n \log n)$ time algorithm for Sorting implies an $O(n \log n)$ time algorithm for Distinct Elements problem.
- If there is *no* $o(n \log n)$ time algorithm for Distinct Elements problem then there is *no* $o(n \log n)$ time algorithm for Sorting.

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Recursion

Reduction: reduce one problem to another

Recursion: a special case of reduction

- reduce problem to a *smaller* instance of *itself*
- self-reduction
- Problem instance of size $\mathbf{n}$ is reduced to one or more instances of size $\mathbf{n-1}$ or less.
- For termination, problem instances of small size are solved by some other method as *base cases*

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Recursion

- Recursion is a very powerful and fundamental technique
- Basis for several other methods
  - Divide and conquer
  - Dynamic programming
  - Enumeration and branch and bound etc
  - Some classes of greedy algorithms
- Makes proof of correctness easy (via induction)
- Recurrences arise in analysis

Selection Sort

Sort a given array $\mathbf{A[1..n]}$ of integers.

Recursive version of Selection sort.

SelectSort($\mathbf{A[1..n]}$):
    if $n = 1$ **return**
    `Find smallest number in` $\mathbf{A}$. `Let` $\mathbf{A[i]}$ `be smallest number`
    `Swap` $\mathbf{A[1]}$ `and` $\mathbf{A[i]}$
    SelectSort($\mathbf{A[2..n]}$)

$\mathbf{T(n)}$: time for **SelectSort** on an $\mathbf{n}$ element array.

$\mathbf{T(n) = T(n-1) + n}$ for $\mathbf{n > 1}$ and $\mathbf{T(1) = 1}$ for $\mathbf{n = 1}$

$\mathbf{T(n) = \Theta(n^2)}$.

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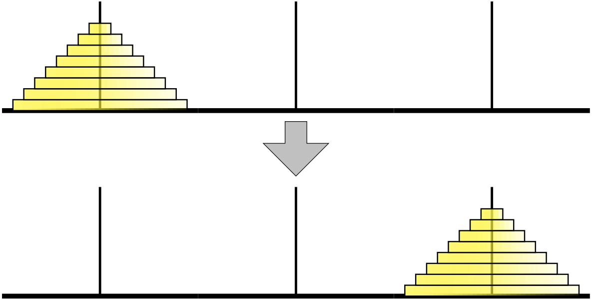
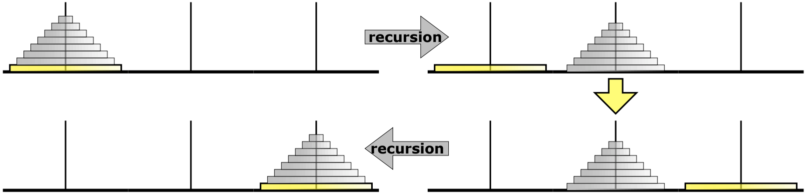
Tower of Hanoi



The Tower of Hanoi puzzle

Move stack of **n** disks from peg **0** to peg **2**, one disk at a time.
Rule: cannot put a larger disk on a smaller disk.
Question: what is a strategy and how many moves does it take?

Tower of Hanoi via Recursion



The Tower of Hanoi algorithm; ignore everything but the bottom disk

Recursive Algorithm

```
Hanoi(n, src, dest, tmp):
    If (n > 0) then
        Hanoi(n − 1, src, tmp, dest)
        Move disk n from src to dest
        Hanoi(n − 1, tmp, dest, src)
```

$\mathbf{T(n)}$: time to move $\mathbf{n}$ disks via recursive strategy

$$T(n) = 2T(n-1) + 1 \qquad n > 1 \qquad \text{and } T(1) = 1$$

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Analysis

$$
\begin{aligned}
T(n) &= 2T(n-1)+1 \\
&= 2^2T(n-2)+2+1 \\
&= \dots \\
&= 2^iT(n-i)+2^{i-1}+2^{i-2}+\dots+1 \\
&= \dots \\
&= 2^{n-1}T(1)+2^{n-2}+\dots+1 \\
&= 2^{n-1}+2^{n-2}+\dots+1 \\
&= (2^n-1)/(2-1) = 2^n-1
\end{aligned}
$$

Non-Recursive Algorithms for Tower of Hanoi

Pegs numbered $\mathbf{0, 1, 2}$

Non-recursive Algorithm 1:

- Always move smallest disk forward if $\mathbf{n}$ is even, backward if $\mathbf{n}$ is odd.
- Never move the same disk twice in a row.
- Done when no legal move.

Non-recursive Algorithm 2:

- Let $\rho(\mathbf{n})$ be the smallest integer $\mathbf{k}$ such that $\mathbf{n}/\mathbf{2^k}$ is *not* an integer. Example: $\rho(40) = 4$, $\rho(18) = 2$.
- In step $\mathbf{i}$ move disk $\rho(\mathbf{i})$ forward if $\mathbf{n} - \mathbf{i}$ is even and backward if $\mathbf{n} - \mathbf{i}$ is odd.

Moves are exactly same as those of recursive algorithm. Prove by induction.

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Part II

Divide and Conquer

Divide and Conquer Paradigm

Divide and Conquer is a common and useful type of recursion

Approach

- Break problem instance into smaller instances - divide step
- Recursively solve problem on smaller instances
- Combine solutions to smaller instances to obtain a solution to the original instance - conquer step

Question: Why is this not plain recursion?

- In divide and conquer, each smaller instance is typically at least a constant factor smaller than the original instance which leads to efficient running times.
- There are many examples of this particular type of recursion that it deserves its own treatment.

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Sorting

Input Given an array of $\mathbf{n}$ elements

Goal Rearrange them in ascending order

Merge Sort [von Neumann]

MergeSort

1. Input: Array $\mathbf{A}[1 \ldots \mathbf{n}]$

   A L G O R I T H M S

2. Divide into subarrays $\mathbf{A}[1 \ldots \mathbf{m}]$ and $\mathbf{A}[\mathbf{m}+1 \ldots \mathbf{n}]$, where $\mathbf{m} = \lfloor \mathbf{n}/2 \rfloor$

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3. Recursively MergeSort $\mathbf{A}[1 \ldots \mathbf{m}]$ and $\mathbf{A}[\mathbf{m}+1 \ldots \mathbf{n}]$

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4. Merge the sorted arrays

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Merging Sorted Arrays

- Use a new array **C** to store the merged array
- Scan **A** and **B** from left-to-right, storing elements in **C** in order

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- Merge two arrays using only constantly more extra space (in-place merge sort): doable but complicated and typically impractical

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Running Time

$\mathbf{T(n)}$: time for merge sort to sort an $\mathbf{n}$ element array

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

What do we want as a solution to the recurrence?

Almost always only an *asymptotically* tight bound. That is we want to know $\mathbf{f(n)}$ such that $\mathbf{T(n) = \Theta(f(n))}$.

- $\mathbf{T(n) = O(f(n))}$ - upper bound
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Solving Recurrences: Some Techniques

- Know some basic math: geometric series, logarithms, exponentials, elementary calculus
- Expand the recurrence and spot a pattern and use simple math
- Recursion tree method — imagine the computation as a tree
- Guess and verify — useful for proving upper and lower bounds even if not tight bounds

Albert Einstein: "Everything should be made as simple as possible, but not simpler."

Know where to be loose in analysis and where to be tight. Comes with practice, practice, practice!

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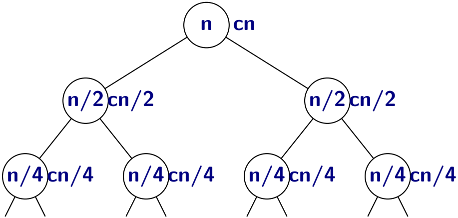

2. Identify a pattern. At the ith level total work is cn
3. Sum over all levels. The number of levels is $\log n$. So total is $cn \log n = O(n \log n)$

Recursion Trees

MergeSort: $\mathbf{n}$ is a power of $\mathbf{2}$

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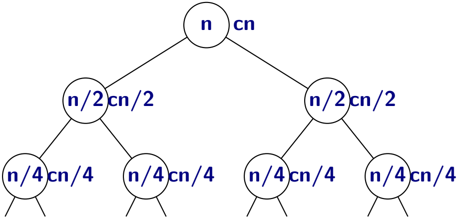

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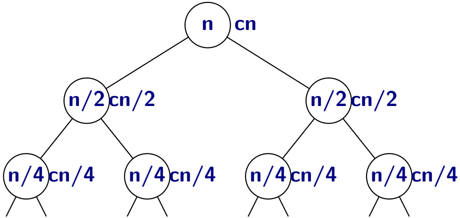

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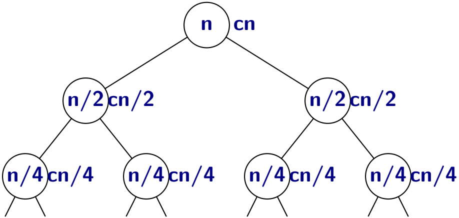

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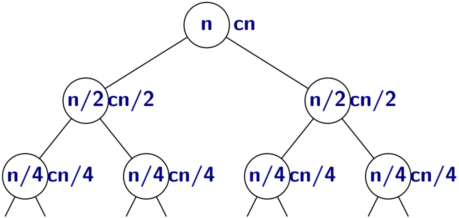

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- When n is not a power of 2, the running time of mergesort is expressed as

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$$

- $n_1 = 2^{k-1} < n \le 2^k = n_2$ (n_1, n_2 powers of 2)
- $T(n_1) < T(n) \le T(n_2)$ (Why?)
- $T(n) = \Theta(n \log n)$ since $n/2 \le n_1 < n \le n_2 \le 2n$.

MergeSort Analysis

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Observation: For any number x, $\lfloor x/2 \rfloor + \lceil x/2 \rceil = x$.

When $\mathbf{n}$ is not a power of $\mathbf{2}$: Guess and Verify

If $\mathbf{n}$ is power of $\mathbf{2}$ we saw that $\mathbf{T(n) = \Theta(n \log n)}$.
Can *guess* that $\mathbf{T(n) = \Theta(n \log n)}$ for all $\mathbf{n}$.
Verify? proof by induction!

Induction Hypothesis: $\mathbf{T(n) \leq 2cn \log n}$ for all $\mathbf{n \geq 1}$
Base Case: $\mathbf{n = 1}$. $\mathbf{T(1) = 0}$ since no need to do any work and $\mathbf{2cn \log n = 0}$ for $\mathbf{n = 1}$.
Induction Step Assume $\mathbf{T(k) \leq 2ck \log k}$ for all $\mathbf{k < n}$ and prove it for $\mathbf{k = n}$.

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Induction Step

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\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn \\
&\leq 2c\lfloor n/2 \rfloor \log \lfloor n/2 \rfloor + 2c\lceil n/2 \rceil \log \lceil n/2 \rceil + cn \quad \text{(by induc} \\
&\leq 2c\lfloor n/2 \rfloor \log \lceil n/2 \rceil + 2c\lceil n/2 \rceil \log \lceil n/2 \rceil + cn \\
&\leq 2c(\lfloor n/2 \rfloor + \lceil n/2 \rceil) \log \lceil n/2 \rceil + cn \\
&\leq 2cn \log \lceil n/2 \rceil + cn \\
&\leq 2cn \log(2n/3) + cn \quad \text{(since } \lceil n/2 \rceil \leq 2n/3 \text{ for all } n \geq 2\text{)} \\
&\leq 2cn \log n + cn(1 - 2\log 3/2) \\
&\leq 2cn \log n + cn(\log 2 - \log 9/4) \\
&\leq 2cn \log n
\end{aligned}
$$

Guess and Verify

The math worked out like magic!
Why was $\mathbf{2cn \log n}$ chosen instead of say $\mathbf{4cn \log n}$?

Typically we don't know upfront what constant to choose. Instead we assume that $\mathbf{T(n)} \leq \alpha \mathbf{cn \log n}$ for some constant α that will be fixed later. All we need to prove that there is some sufficiently large constant α that will make the algebra go through.

We need to choose α such that $\alpha \log 3/2 > 1$.

Typically you do the algebra with α and then show at the end that α can be chosen to be sufficiently large constant.

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Guess and Verify: When is a guess incorrect?

Suppose we guessed that the soln to the mergesort recurrent is $\mathbf{T(n) = O(n)}$. We try to prove by induction that $\mathbf{T(n) \leq \alpha cn}$ for some constant α.

Induction Step: attempt

$$\begin{aligned} \mathbf{T(n)} &= \mathbf{T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn} \\ &\leq \mathbf{\alpha c \lfloor n/2 \rfloor + \alpha c \lceil n/2 \rceil + cn} \\ &\leq \mathbf{\alpha cn + cn} \\ &\leq \mathbf{(\alpha + 1)cn} \end{aligned}$$

But we want to show that $\mathbf{T(n) \leq \alpha cn}$! So guess does not work for *any* constant α. Suggests that our guess is incorrect.

Selection Sort vs Merge Sort

- Selection Sort spends $\mathbf{O(n)}$ work to reduce problem from $\mathbf{n}$ to $\mathbf{n-1}$ leading to $\mathbf{O(n^2)}$ running time.
- Merge Sort spends $\mathbf{O(n)}$ time *after* reducing problem to two instances of size $\mathbf{n/2}$ each. Running time is $\mathbf{O(n \log n)}$

Question: Merge Sort splits into 2 (roughly) equal sized arrays. Can we do better by splitting into more than 2 arrays? Say $\mathbf{k}$ arrays of size $\mathbf{n/k}$ each?

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Quick Sort[Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself. Linear scan of array does it. Time is $O(n)$
3. Recursively sort the subarrays, and concatenate them.

Example:

- array: 16, 12, 14, 20, 5, 3, 18, 19, 1
- pivot: 16
- split into 12, 14, 5, 3, 1 and 20, 19, 18 and recursively sort
- put them together with pivot in middle

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Time Analysis

- Let $\mathbf{k}$ be the rank of the chosen pivot. Then, $\mathbf{T(n) = T(k-1) + T(n-k) + O(n)}$
- If $\mathbf{k} = \lceil \mathbf{n}/2 \rceil$ then $\mathbf{T(n) = T(\lceil n/2 \rceil - 1) + T(\lfloor n/2 \rfloor) + O(n) \leq 2T(n/2) + O(n)}$. Then, $\mathbf{T(n) = O(n \log n)}$.
  - Theoretically, median can be found in linear time.
- Typically, pivot is the first or last element of array. Then,

$$\mathbf{T(n) = \max_{1 \leq k \leq n} (T(k-1) + T(n-k) + O(n))}$$

In the worst case $\mathbf{T(n) = T(n-1) + O(n)}$, which means $\mathbf{T(n) = O(n^2)}$. Happens if array is already sorted and pivot is always first element.

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Part III

Fast Multiplication

Multiplying Numbers

Problem Given two $\mathbf{n}$-digit numbers $\mathbf{x}$ and $\mathbf{y}$, compute their product.

Grade School Multiplication

Compute "partial product" by multiplying each digit of $\mathbf{y}$ with $\mathbf{x}$ and adding the partial products.

$$
\begin{array}{r}
3141 \\
\times 2718 \\
\hline
25128 \\
3141 \\
21987 \\
6282 \\
\hline
8537238
\end{array}
$$

Time Analysis of Grade School Multiplication

- Each partial product: $\mathbf{\Theta(n)}$
- Number of partial products: $\mathbf{\Theta(n)}$
- Addition of partial products: $\mathbf{\Theta(n^2)}$
- Total time: $\mathbf{\Theta(n^2)}$

A Trick of Gauss

Carl Fridrich Gauss: 1777–1855 "Prince of Mathematicians"

Observation: Multiply two complex numbers: $(a + bi)$ and $(c + di)$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

How many multiplications do we need?

Only 3! If we do extra additions and subtractions.
Compute $ac, bd, (a + b)(c + d)$. Then
$(ad + bc) = (a + b)(c + d) - ac - bd$

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Divide and Conquer

Assume $\mathbf{n}$ is a power of $\mathbf{2}$ for simplicity and numbers are in decimal.

- $x = x_{n-1}x_{n-2}\ldots x_0$ and $y = y_{n-1}y_{n-2}\ldots y_0$
- $x = 10^{n/2}x_L + x_R$ where $x_L = x_{n-1}\ldots x_{n/2}$ and $x_R = x_{n/2-1}\ldots x_0$
- $y = 10^{n/2}y_L + y_R$ where $y_L = y_{n-1}\ldots y_{n/2}$ and $y_R = y_{n/2-1}\ldots y_0$

Therefore

$$xy = (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) = 10^n x_L y_L + 10^{n/2}(x_L y_R + x_R y_L) + x$$

Example

$$
\begin{aligned}
1234 \times 5678 &= (100 \times 12 + 34) \times (100 \times 56 + 78) \\
&= 10000 \times 12 \times 56 \\
&\quad +100 \times (12 \times 78 + 34 \times 56) \\
&\quad +34 \times 78
\end{aligned}
$$

Time Analysis

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4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

$T(n) = \Theta(n^2)$. No better than grade school multiplication!

Can we invoke Gauss's trick here?

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Gauss trick: $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

Recursively compute only $x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R)$.

Time Analysis

Running time is given by

$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

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State of the Art

Schönhage-Strassen 1971: $\mathbf{O(n \log n \log \log n)}$ time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007: $\mathbf{O(n \log n 2^{O(\log^* n)})}$ time

Conjecture: there is an $\mathbf{O(n \log n)}$ time algorithm

Analyzing the Recurrences

- Basic divide and conquer: $T(n) = 4T(n/2) + O(n)$, $T(1) = 1$. **Claim:** $T(n) = \Theta(n^2)$.
- Saving a multiplication: $T(n) = 3T(n/2) + O(n)$, $T(1) = 1$. **Claim:** $T(n) = \Theta(n^{1+\log 1.5})$

Use recursion tree method:

- In both cases, depth of recursion $L = \log n$.
- Work at depth i is $4^i n/2^i$ and $3^i n/2^i$ respectively: number of children at depth i times the work at each child
- Total work is therefore $n \sum_{i=0}^{L} 2^i$ and $n \sum_{i=0}^{L} (3/2)^i$ respectively.

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Recursion tree analysis

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