3.1 Breadth First Search

3.1.0.1 Breadth First Search (BFS)

Overview

1. **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.

2. It processes the vertices in the graph in the order of their shortest distance from the vertex \( s \) (the start vertex).

As such...

(A) **DFS** good for exploring graph structure
(B) **BFS** good for exploring **distances**

3.1.0.2 Queue Data Structure

Queues

A **queue** is a list of elements which supports the following operations
(A) **enqueue**: Adds an element to the end of the list
(B) **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.
3.1.0.3 BFS Algorithm

Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

<table>
<thead>
<tr>
<th>BFS(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark all vertices as unvisited</td>
</tr>
<tr>
<td>Initialize search tree $T$ to be empty</td>
</tr>
<tr>
<td>Mark vertex $s$ as visited</td>
</tr>
<tr>
<td>set $Q$ to be the empty queue</td>
</tr>
<tr>
<td>enq(s)</td>
</tr>
<tr>
<td>while $Q$ is nonempty do</td>
</tr>
<tr>
<td>$u = \text{deq}(Q)$</td>
</tr>
<tr>
<td>for each vertex $v \in \text{Adj}(u)$</td>
</tr>
<tr>
<td>if $v$ is not visited then</td>
</tr>
<tr>
<td>add edge $(u, v)$ to $T$</td>
</tr>
<tr>
<td>Mark $v$ as visited and enq(v)</td>
</tr>
</tbody>
</table>

Proposition 3.1.1 BFS($s$) runs in $O(n + m)$ time.

3.1.0.4 BFS: An Example in Undirected Graphs


BFS tree is the set of black edges.
3.1.0.5 BFS: An Example in Directed Graphs

3.1.0.6 BFS with Distance

BFS\( (s) \)

Mark all vertices as unvisited and for each \( v \) set \( \text{dist}(v) = \infty \)

Initialize search tree \( T \) to be empty

Mark vertex \( s \) as visited and set \( \text{dist}(s) = 0 \)

set \( Q \) to be the empty queue

\( \text{enq}(s) \)

while \( Q \) is nonempty do

\( u = \text{deq}(Q) \)

for each vertex \( v \in \text{Adj}(u) \) do

if \( v \) is not visited do

add edge \( (u, v) \) to \( T \)

Mark \( v \) as visited, \( \text{enq}(v) \) and set \( \text{dist}(v) = \text{dist}(u) + 1 \)

3.1.0.7 Properties of BFS: Undirected Graphs

Proposition 3.1.2 The following properties hold upon termination of BFS\( (s) \):

1. The search tree contains exactly the set of vertices in the connected component of \( s \).

2. If \( \text{dist}(u) < \text{dist}(v) \) then \( u \) is visited before \( v \).

3. For every vertex \( u \), \( \text{dist}(u) \) is indeed the length of shortest path from \( s \) to \( u \).

4. If \( u, v \) are in connected component of \( s \) and \( e = \{u, v\} \) is an edge of \( G \), then either \( e \) is an edge in the search tree, or \( |\text{dist}(u) - \text{dist}(v)| \leq 1 \).

Proof: Exercise.

3.1.0.8 Properties of BFS: Directed Graphs

Proposition 3.1.3 The following properties hold upon termination of BFS\( (s) \):

1. The search tree contains exactly the set of vertices reachable from \( s \)
2. If \( \text{dist}(u) < \text{dist}(v) \) then \( u \) is visited before \( v \)

3. For every vertex \( u \), \( \text{dist}(u) \) is indeed the length of shortest path from \( s \) to \( u \)

4. If \( u \) is reachable from \( s \) and \( e = (u, v) \) is an edge of \( G \), then either \( e \) is an edge in the search tree, or \( \text{dist}(v) - \text{dist}(u) \leq 1 \). Not necessarily the case that \( \text{dist}(u) - \text{dist}(v) \leq 1 \).

**Proof**: Exercise.

### 3.1.0.9 BFS with Layers

**BFSLayers**:\( s \):

Mark all vertices as unvisited and initialize \( T \) to be empty
Mark \( s \) as visited and set \( L_0 = \{ s \} \)

\( i = 0 \)

while \( L_i \) is not empty do

initialize \( L_{i+1} \) to be an empty list

for each \( u \) in \( L_i \) do

for each edge \( (u, v) \in \text{Adj}(u) \) do

if \( v \) is not visited

mark \( v \) as visited

add \( (u, v) \) to tree \( T \)

add \( v \) to \( L_{i+1} \)

\( i = i + 1 \)

Running time: \( O(n + m) \)

### 3.1.0.10 Example

### 3.1.0.11 BFS with Layers: Properties

**Proposition 3.1.4** The following properties hold on termination of **BFSLayers**\( (s) \).

(A) **BFSLayers**\( (s) \) outputs a BFS tree

(B) \( L_i \) is the set of vertices at distance exactly \( i \) from \( s \)

(C) If \( G \) is undirected, each edge \( e = \{u, v\} \) is one of three types:

(A) tree edge between two consecutive layers

(B) non-tree forward/backward edge between two consecutive layers

(C) non-tree cross-edge with both \( u, v \) in same layer

(D) \( \Rightarrow \) Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.
3.1.1 BFS with Layers: Properties

3.1.1.1 For directed graphs

Proposition 3.1.5 The following properties hold on termination of BFS$_{Layers}(s)$, if $G$ is directed.

For each edge $e = (u, v)$ is one of four types:

(A) a tree edge between consecutive layers, $u \in L_i$, $v \in L_{i+1}$ for some $i \geq 0$
(B) a non-tree forward edge between consecutive layers
(C) a non-tree backward edge
(D) a cross-edge with both $u, v$ in same layer

3.2 Bipartite Graphs and an application of BFS

3.2.0.2 Bipartite Graphs

Definition 3.2.1 (Bipartite Graph) Undirected graph $G = (V, E)$ is a bipartite graph if $V$ can be partitioned into $X$ and $Y$ s.t. all edges in $E$ are between $X$ and $Y$.

3.2.0.3 Bipartite Graph Characterization

Question
When is a graph bipartite?

Proposition 3.2.2 Every tree is a bipartite graph.

Proof: Root tree $T$ at some node $r$. Let $L_i$ be all nodes at level $i$, that is, $L_i$ is all nodes at distance $i$ from root $r$. Now define $X$ to be all nodes at even levels and $Y$ to be all nodes at odd level. Only edges in $T$ are between levels.

Proposition 3.2.3 An odd length cycle is not bipartite.
3.2.0.4 Odd Cycles are not Bipartite

**Proposition 3.2.4** An odd length cycle is not bipartite.

*Proof*: Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose $C$ is a bipartite graph and let $X, Y$ be the bipartition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if $i$ is odd $u_i \in Y$ if $i$ is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to $X$.

3.2.0.5 Subgraphs

**Definition 3.2.5** Given a graph $G = (V, E)$ a subgraph of $G$ is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

**Proposition 3.2.6** If $G$ is bipartite then any subgraph $H$ of $G$ is also bipartite.

**Proposition 3.2.7** A graph $G$ is not bipartite if $G$ has an odd cycle $C$ as a subgraph.

*Proof*: If $G$ is bipartite then since $C$ is a subgraph, $C$ is also bipartite (by above proposition). However, $C$ is not bipartite!

3.2.0.6 Bipartite Graph Characterization

**Theorem 3.2.8** A graph $G$ is bipartite if and only if it has no odd length cycle as subgraph.

*Proof*: *Only If*: $G$ has an odd cycle implies $G$ is not bipartite.  

*If*: $G$ has no odd length cycle. Assume without loss of generality that $G$ is connected.  

(A) Pick $u$ arbitrarily and do BFS$(u)$  

(B) $X = \bigcup_{i \text{ is even}} L_i$ and $Y = \bigcup_{i \text{ is odd}} L_i$  

(C) **Claim**: $X$ and $Y$ is a valid bipartition if $G$ has no odd length cycle.

3.2.0.7 Proof of Claim

**Claim 3.2.9** In BFS$(u)$ if $a, b \in L_i$ and $(a, b)$ is an edge then there is an odd length cycle containing $(a, b)$.

*Proof*: Let $v$ be least common ancestor of $a, b$ in BFS tree $T$.  
$v$ is in some level $j < i$ (could be $u$ itself).  
Path from $v \leadsto a$ in $T$ is of length $j - i$.  
Path from $v \leadsto b$ in $T$ is of length $j - i$.  
These two paths plus $(a, b)$ forms an odd cycle of length $2(j - i) + 1$.

**Corollary 3.2.10** There is an $O(n+m)$ time algorithm to check if $G$ is bipartite and output an odd cycle if it is not.
3.3 Shortest Paths and Dijkstra’s Algorithm

3.3.0.8 Shortest Path Problems

Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

(A) Given nodes $s, t$ find shortest path from $s$ to $t$.
(B) Given node $s$ find shortest path from $s$ to all other nodes.
(C) Find shortest paths for all pairs of nodes.

Many applications!

3.3.0.9 Single-Source Shortest Paths: Non-Negative Edge Lengths

Single-Source Shortest Path Problems

Input A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

(A) Given nodes $s, t$ find shortest path from $s$ to $t$.
(B) Given node $s$ find shortest path from $s$ to all other nodes.

(A) Restrict attention to directed graphs
(B) Undirected graph problem can be reduced to directed graph problem - how?
   (A) Given undirected graph $G$, create a new directed graph $G'$ by replacing each edge
       $\{u, v\}$ in $G$ by $(u, v)$ and $(v, u)$ in $G'$.
   (B) set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$
   (C) Exercise: show reduction works

3.3.0.10 Single-Source Shortest Paths via BFS

Special case: All edge lengths are 1.
(A) Run BFS$(s)$ to get shortest path distances from $s$ to all other nodes.
(B) $O(m + n)$ time algorithm.

Special case: Suppose $\ell(e)$ is an integer for all $e$?
Can we use BFS? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on $e$.

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. BFS takes $O(mL + n)$ time. Not efficient if $L$ is large.
3.3.0.11 Towards an algorithm

Why does BFS work?

BFS(s) explores nodes in increasing distance from s

**Lemma 3.3.1** Let $G$ be a directed graph with non-negative edge lengths. Let $\text{dist}(s, v)$ denote the shortest path length from $s$ to $v$. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

(A) $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$

(B) $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$.

**Proof**: Suppose not. Then for some $i < k$ there is a path $P'$ from $s$ to $v_i$ of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then $P'$ concatenated with $v_i \rightarrow v_{i+1} \ldots \rightarrow v_k$ contains a strictly shorter path to $v_k$ than $s = v_0 \rightarrow v_1 \ldots \rightarrow v_k$.

3.3.0.12 A proof by picture

![Diagram of shortest path from $v_0$ to $v_6$](image)

3.3.0.13 A Basic Strategy

Explore vertices in increasing order of distance from $s$:

(For simplicity assume that nodes are at different distances from $s$ and that no edge has zero length)
Initialize for each node \( v \), \( \text{dist}(s, v) = \infty \)
Initialize \( S = \emptyset \),
for \( i = 1 \) to \( |V| \) do
  (* Invariant: \( S \) contains the \( i-1 \) closest nodes to \( s \) *)
  Among nodes in \( V \setminus S \), find the node \( v \) that is the \( i \)th closest to \( s \)
  Update \( \text{dist}(s, v) \)
  \( S = S \cup \{ v \} \)

How can we implement the step in the for loop?

### 3.3.0.14 Finding the \( i \)th closest node

(A) \( S \) contains the \( i-1 \) closest nodes to \( s \)
(B) Want to find the \( i \)th closest node from \( V - S \).

What do we know about the \( i \)th closest node?

Claim 3.3.2 Let \( P \) be a shortest path from \( s \) to \( v \) where \( v \) is the \( i \)th closest node. Then, all intermediate nodes in \( P \) belong to \( S \).

Proof: If \( P \) had an intermediate node \( u \) not in \( S \) then \( u \) will be closer to \( s \) than \( v \). Implies \( v \) is not the \( i \)th closest node to \( s \) - recall that \( S \) already has the \( i-1 \) closest nodes.

### 3.3.1 Finding the \( i \)th closest node repeatedly

#### 3.3.1.1 An example

![Graph Example](image)

#### 3.3.1.2 Finding the \( i \)th closest node

Corollary 3.3.3 The \( i \)th closest node is adjacent to \( S \).
3.3.1.3 Finding the $i$th closest node

(A) $S$ contains the $i - 1$ closest nodes to $s$
(B) Want to find the $i$th closest node from $V - S$.
(A) For each $u \in V - S$ let $P(s, u, S)$ be a shortest path from $s$ to $u$ using only nodes in $S$ as intermediate vertices.
(B) Let $d'(s, u)$ be the length of $P(s, u, S)$

Observations: for each $u \in V - S$,
(A) $\text{dist}(s, u) \leq d'(s, u)$ since we are constraining the paths
(B) $d'(s, u) = \min_{a \in S}(\text{dist}(s, a) + \ell(a, u))$ - Why?

Lemma 3.3.4 If $v$ is the $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

3.3.1.4 Finding the $i$th closest node

Lemma 3.3.5 If $v$ is an $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

Proof: Let $v$ be the $i$th closest node to $s$. Then there is a shortest path $P$ from $s$ to $v$ that contains only nodes in $S$ as intermediate nodes (see previous claim). Therefore $d'(s, v) = \text{dist}(s, v)$.

3.3.1.5 Finding the $i$th closest node

Lemma 3.3.6 If $v$ is an $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

Corollary 3.3.7 The $i$th closest node to $s$ is the node $v \in V - S$ such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$.

Proof: For every node $u \in V - S$, $\text{dist}(s, u) \leq d'(s, u)$ and for the $i$th closest node $v$, $\text{dist}(s, v) = d'(s, v)$. Moreover, $\text{dist}(s, u) \geq \text{dist}(s, v)$ for each $u \in V - S$.
3.3.1.6 Algorithm

```plaintext
Initialize for each node v: \( \text{dist}(s,v) = \infty \)
Initialize \( S = \emptyset \), \( d'(s,s) = 0 \)
for \( i = 1 \) to \( |V| \) do
  (* Invariant: \( S \) contains the \( i-1 \) closest nodes to \( s \) *)
  (* Invariant: \( d'(s,u) \) is shortest path distance from \( u \) to \( s 
  
  using only \( S \) as intermediate nodes*)
  Let \( v \) be such that \( d'(s,v) = \min_{u \in V - S} d'(s,u) \)
  \( \text{dist}(s,v) = d'(s,v) \)
  \( S = S \cup \{v\} \)
  for each node \( u \) in \( V \setminus S \)
    compute \( d'(s,u) = \min_{a \in S} (\text{dist}(s,a) + \ell(a,u)) \)
```

**Correctness:** By induction on \( i \) using previous lemmas.

**Running time:** \( O(n \cdot (n + m)) \) time.

(A) \( n \) outer iterations. In each iteration, \( d'(s,u) \) for each \( u \) by scanning all edges out of
nodes in \( S \); \( O(m + n) \) time/iteration.

3.3.1.7 Example

3.3.1.8 Improved Algorithm

(A) Main work is to compute the \( d'(s,u) \) values in each iteration
(B) \( d'(s,u) \) changes from iteration \( i \) to \( i + 1 \) only because of the node \( v \) that is added to \( S \)
in iteration \( i \).
Initialize for each node \( v \), \( \text{dist}(s, v) = d'(s, v) = \infty \)
Initialize \( S = \emptyset \), \( d'(s, s) = 0 \)

for \( i = 1 \) to \( |V| \) do

// \( S \) contains the \( i-1 \) closest nodes to \( s \),
// and the values of \( d'(s, u) \) are current
Let \( v \) be such that \( d'(s, v) = \min_{u \in V-S} d'(s, u) \)
\( \text{dist}(s, v) = d'(s, v) \)
\( S = S \cup \{v\} \)
Update \( d'(s, u) \) for each \( u \) in \( V-S \) as follows:
\( d'(s, u) = \min(d'(s, u), \text{dist}(s, v) + \ell(v, u)) \)

**Running time:** \( O(m + n^2) \) time.
(A) \( n \) outer iterations and in each iteration following steps
(B) updating \( d'(s, u) \) after \( v \) added takes \( O(deg(v)) \) time so total work is \( O(m) \) since a node enters \( S \) only once
(C) Finding \( v \) from \( d'(s, u) \) values is \( O(n) \) time

### 3.3.1.9 Dijkstra’s Algorithm

(A) eliminate \( d'(s, u) \) and let \( \text{dist}(s, u) \) maintain it
(B) update \( \text{dist} \) values after adding \( v \) by scanning edges out of \( v \)

Priority Queues to maintain \( \text{dist} \) values for faster running time

(A) Using heaps and standard priority queues: \( O((m + n) \log n) \)
(B) Using Fibonacci heaps: \( O(m + n \log n) \).

### 3.3.2 Priority Queues

#### 3.3.2.1 Priority Queues

Data structure to store a set \( S \) of \( n \) elements where each element \( v \in S \) has an associated real/integer key \( k(v) \) such that the following operations
(A) \text{makeQ}: create an empty queue
(B) \text{findMin}: find the minimum key in \( S \)
(C) \text{extractMin}: Remove \( v \in S \) with smallest key and return it
(D) \text{add}(v, k(v))$: Add new element \( v \) with key \( k(v) \) to \( S \)
(E) \text{delete}(v)$: Remove element \( v \) from \( S \)
(F) \texttt{decreaseKey}(v, k'(v)): decrease key of v from \(k(v)\) (current key) to \(k'(v)\) (new key). 
Assumption: \(k'(v) \leq k(v)\)

(G) \texttt{meld}: merge two separate priority queues into one 
can be performed in \(O(\log n)\) time each.
\texttt{decreaseKey} via \texttt{delete} and \texttt{add}

### 3.3.2.2 Dijkstra’s Algorithm using Priority Queues

```plaintext
Q = makePQ()
insert(Q, (s, 0))
for each node \(u \neq s\) do
    insert(Q, (u, \infty))
S = \emptyset
for \(i = 1\) to \(|V|\) do
    (v, dist(s, v)) = extractMin(Q)
    S = S \cup \{v\}
    For each \(u\) in \text{Adj}(v) do
        decreaseKey(Q, (u, \min(dist(s, u), dist(s, v) + \ell(v, u))))
```

Priority Queue operations:
(A) \(O(n)\) \texttt{insert} operations
(B) \(O(n)\) \texttt{extractMin} operations
(C) \(O(m)\) \texttt{decreaseKey} operations

### 3.3.2.3 Implementing Priority Queues via Heaps

#### Using Heaps
Store elements in a heap based on the key value
(A) All operations can be done in \(O(\log n)\) time

Dijkstra’s algorithm can be implemented in \(O((n + m) \log n)\) time.

### 3.3.2.4 Priority Queues: Fibonacci Heaps/Relaxed Heaps

#### Fibonacci Heaps
(A) \texttt{extractMin}, \texttt{add}, \texttt{delete}, \texttt{meld} in \(O(\log n)\) time 
(B) \texttt{decreaseKey} in \(O(1)\) \texttt{amortized} time: \(\ell\) \texttt{decreaseKey} operations for \(\ell \geq n\) take \texttt{together} \(O(\ell)\) time 
(C) Relaxed Heaps: \texttt{decreaseKey} in \(O(1)\) worst case time but at the expense of \texttt{meld} (not necessary for Dijkstra’s algorithm)

— Dijkstra’s algorithm can be implemented in \(O(n \log n + m)\) time. If \(m = \Omega(n \log n)\), running time is linear in input size.