Breadth First Search, Dijkstra’s Algorithm for Shortest Paths

Lecture 3
January 25, 2011
Part I

Breadth First Search
Breadth First Search (BFS)

Overview

(A) **BFS** is obtained from **BasicSearch** by processing edges using a data structure called a **queue**.
(B) It processes the vertices in the graph in the order of their shortest distance from the vertex **s** (the start vertex).

As such...

- **DFS** good for exploring graph structure
- **BFS** good for exploring *distances*
Queues

A **queue** is a list of elements which supports the following operations:

- **enqueue**: Adds an element to the end of the list
- **dequeue**: Removes an element from the front of the list

Elements are extracted in **first-in first-out (FIFO)** order, i.e., elements are picked in the order in which they were inserted.
BFS Algorithm

Given (undirected or directed) graph $G = (V, E)$ and node $s \in V$

**BFS(s)**
- Mark all vertices as unvisited
- Initialize search tree $T$ to be empty
- Mark vertex $s$ as visited
- set $Q$ to be the empty queue

**enq(s)**

**while** $Q$ is nonempty **do**

- $u = \text{deq}(Q)$
- **for** each vertex $v \in \text{Adj}(u)$
  - **if** $v$ is not visited **then**
    - add edge $(u, v)$ to $T$
    - Mark $v$ as visited and $\text{enq}(v)$

**Proposition**

**BFS(s)** runs in $O(n + m)$ time.
BFS: An Example in Undirected Graphs

BFS tree is the set of black edges.

1. [1] 4. [4,5,7,8]
2. [2,3] 5. [5,7,8]
3. [3,4,5] 6. [7,8,6]
4. [4,5,7,8] 5. [5,7,8]
9. []
BFS: An Example in Undirected Graphs

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BFS: An Example in Undirected Graphs

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3. \([3,4,5]\)
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7. \([8,6]\)
8. \([6]\)
9. \([]\)

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BFS: An Example in Undirected Graphs

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7. [8,6]
8. [6]
9. []

BFS tree is the set of black edges.
BFS: An Example in Directed Graphs

A directed graph (also called a digraph) is $G = (V, E)$, where $V$ is a set of vertices or nodes and $E \subseteq V \times V$ is the set of ordered pairs of vertices called edges.
BFS with Distance

**BFS(s)**

Mark all vertices as unvisited and for each \( v \) set \( \text{dist}(v) = \infty \)

Initialize search tree \( T \) to be empty

Mark vertex \( s \) as visited and set \( \text{dist}(s) = 0 \)

set \( Q \) to be the empty queue

\textbf{enq}(s)

\textbf{while} \( Q \) is nonempty \textbf{do}

\( u = \text{deq}(Q) \)

\textbf{for} each vertex \( v \in \text{Adj}(u) \) \textbf{do}

\textbf{if} \( v \) is not visited \textbf{do}

add edge \((u, v)\) to \( T \)

Mark \( v \) as visited, \textbf{enq}(v)

and set \( \text{dist}(v) = \text{dist}(u) + 1 \)
Properties of BFS: Undirected Graphs

Proposition

The following properties hold upon termination of BFS\( (s) \)

(A) The search tree contains exactly the set of vertices in the connected component of \( s \).

(B) If \( \text{dist}(u) < \text{dist}(v) \) then \( u \) is visited before \( v \).

(C) For every vertex \( u \), \( \text{dist}(u) \) is indeed the length of shortest path from \( s \) to \( u \).

(D) If \( u, v \) are in connected component of \( s \) and \( e = \{u, v\} \) is an edge of \( G \), then either \( e \) is an edge in the search tree, or \( |\text{dist}(u) − \text{dist}(v)| \leq 1 \).

Proof.

Exercise.
Properties of BFS: Directed Graphs

Proposition

The following properties hold upon termination of $\text{BFS}(s)$:

(A) The search tree contains exactly the set of vertices reachable from $s$

(B) If $\text{dist}(u) < \text{dist}(v)$ then $u$ is visited before $v$

(C) For every vertex $u$, $\text{dist}(u)$ is indeed the length of shortest path from $s$ to $u$

(D) If $u$ is reachable from $s$ and $e = (u, v)$ is an edge of $G$, then either $e$ is an edge in the search tree, or $\text{dist}(v) - \text{dist}(u) \leq 1$. Not necessarily the case that $\text{dist}(u) - \text{dist}(v) \leq 1$.

Proof.

Exercise.
BFS with Layers

\textbf{BFSLayers}(s):
Mark all vertices as unvisited and initialize $T$ to be empty
Mark $s$ as visited and set $L_0 = \{s\}$

$i = 0$

\textbf{while} $L_i$ is not empty \textbf{do}

\hspace{1em} initialize $L_{i+1}$ to be an empty list

\hspace{1em} \textbf{for} each $u$ in $L_i$ \textbf{do}

\hspace{2em} \textbf{for} each edge $(u, v) \in \text{Adj}(u)$ \textbf{do}

\hspace{3em} \textbf{if} $v$ is not visited

\hspace{4em} mark $v$ as visited

\hspace{4em} add $(u, v)$ to tree $T$

\hspace{3em} add $v$ to $L_{i+1}$

\hspace{1em} $i = i + 1$

\textbf{Running time:} $O(n + m)$
BFS with Layers

**BFS\text{Layers}(s):**

Mark all vertices as unvisited and initialize $T$ to be empty
Mark $s$ as visited and set $L_0 = \{s\}$
i = 0

while $L_i$ is not empty do

initialize $L_{i+1}$ to be an empty list

for each $u$ in $L_i$ do

for each edge $(u, v) \in \text{Adj}(u)$ do

if $v$ is not visited

mark $v$ as visited

add $(u, v)$ to tree $T$

add $v$ to $L_{i+1}$

$i = i + 1$

Running time: $O(n + m)$
Example
BFS with Layers: Properties

Proposition

The following properties hold on termination of $\text{BFSLayers}(s)$.

- $\text{BFSLayers}(s)$ outputs a BFS tree
- $L_i$ is the set of vertices at distance exactly $i$ from $s$
- If $G$ is undirected, each edge $e = \{u, v\}$ is one of three types:
  - tree edge between two consecutive layers
  - non-tree forward/backward edge between two consecutive layers
  - non-tree cross-edge with both $u, v$ in same layer
- $\iff$ Every edge in the graph is either between two vertices that are either (i) in the same layer, or (ii) in two consecutive layers.
BFS with Layers: Properties
For directed graphs

**Proposition**

The following properties hold on termination of \textbf{BFSLayers}(s), if \(G\) is directed.

For each edge \(e = (u, v)\) is one of four types:
- a **tree** edge between consecutive layers, \(u \in L_i, v \in L_{i+1}\) for some \(i \geq 0\)
- a non-tree **forward** edge between consecutive layers
- a non-tree **backward** edge
- a **cross-edge** with both \(u, v\) in same layer
Part II

Bipartite Graphs and an application of BFS
Bipartite Graphs

Definition (Bipartite Graph)

Undirected graph $G = (V, E)$ is a bipartite graph if $V$ can be partitioned into $X$ and $Y$ s.t. all edges in $E$ are between $X$ and $Y$. 
Question

When is a graph bipartite?

Proposition

Every tree is a bipartite graph.

Proof.

Root tree $T$ at some node $r$. Let $L_i$ be all nodes at level $i$, that is, $L_i$ is all nodes at distance $i$ from root $r$. Now define $X$ to be all nodes at even levels and $Y$ to be all nodes at odd level. Only edges in $T$ are between levels.

Proposition

An odd length cycle is not bipartite.
Question
When is a graph bipartite?

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An odd length cycle is not bipartite.
Proposition

An odd length cycle is not bipartite.

Proof.

Let $C = u_1, u_2, \ldots, u_{2k+1}, u_1$ be an odd cycle. Suppose $C$ is a bipartite graph and let $X, Y$ be the bipartition. Without loss of generality $u_1 \in X$. Implies $u_2 \in Y$. Implies $u_3 \in X$. Inductively, $u_i \in X$ if $i$ is odd $u_i \in Y$ if $i$ is even. But $\{u_1, u_{2k+1}\}$ is an edge and both belong to $X$!
Subgraphs

Definition

Given a graph $G = (V, E)$ a subgraph of $G$ is another graph $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

Proposition

If $G$ is bipartite then any subgraph $H$ of $G$ is also bipartite.

Proposition

A graph $G$ is not bipartite if $G$ has an odd cycle $C$ as a subgraph.

Proof.

If $G$ is bipartite then since $C$ is a subgraph, $C$ is also bipartite (by above proposition). However, $C$ is not bipartite!
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A graph \( G \) is not bipartite if \( G \) has an odd cycle \( C \) as a subgraph.

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A graph \( G \) is not bipartite if \( G \) has an odd cycle \( C \) as a subgraph.

Proof.
If \( G \) is bipartite then since \( C \) is a subgraph, \( C \) is also bipartite (by above proposition). However, \( C \) is not bipartite!
Theorem

A graph $G$ is bipartite if and only if it has no odd length cycle as subgraph.

Proof.

**Only If:** $G$ has an odd cycle implies $G$ is not bipartite.

**If:** $G$ has no odd length cycle. Assume without loss of generality that $G$ is connected.

- Pick $u$ arbitrarily and do $\text{BFS}(u)$
- $X = \bigcup_{i \text{ is even}} L_i$ and $Y = \bigcup_{i \text{ is odd}} L_i$
- **Claim:** $X$ and $Y$ is a valid bipartition if $G$ has no odd length cycle.
Theorem

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Proof.

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- Pick $u$ arbitrarily and do $\text{BFS}(u)$
- $X = \bigcup_{i \text{ is even}} L_i$ and $Y = \bigcup_{i \text{ is odd}} L_i$

**Claim:** $X$ and $Y$ is a valid bipartition if $G$ has no odd length cycle.
Proof of Claim

Claim

In $\text{BFS}(u)$ if $a, b \in L_i$ and $(a, b)$ is an edge then there is an odd length cycle containing $(a, b)$.

Proof.

Let $v$ be least common ancestor of $a, b$ in $\text{BFS}$ tree $T$. $v$ is in some level $j < i$ (could be $u$ itself).
Path from $v \leadsto a$ in $T$ is of length $j - i$.
Path from $v \leadsto b$ in $T$ is of length $j - i$.
These two paths plus $(a, b)$ forms an odd cycle of length $2(j - i) + 1$.

Corollary

There is an $O(n + m)$ time algorithm to check if $G$ is bipartite and output an odd cycle if it is not.
Proof of Claim

Claim

In BFS(u) if \(a, b \in L_i\) and \((a, b)\) is an edge then there is an odd length cycle containing \((a, b)\).

Proof.

Let \(v\) be least common ancestor of \(a, b\) in BFS tree \(T\).
\(v\) is in some level \(j < i\) (could be \(u\) itself).
Path from \(v \rightarrow a\) in \(T\) is of length \(j - i\).
Path from \(v \rightarrow b\) in \(T\) is of length \(j - i\).
These two paths plus \((a, b)\) forms an odd cycle of length 
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There is an \(O(n + m)\) time algorithm to check if \(G\) is bipartite and output an odd cycle if it is not.
Part III

Shortest Paths and Dijkstra’s Algorithm
Shortest Path Problems

Input: A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

- Given nodes $s, t$ find shortest path from $s$ to $t$.
- Given node $s$ find shortest path from $s$ to all other nodes.
- Find shortest paths for all pairs of nodes.

Many applications!
Shortest Path Problems

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Many applications!
Single-Source Shortest Paths: Non-Negative Edge Lengths

**Input** A (undirected or directed) graph \( G = (V, E) \) with non-negative edge lengths. For edge \( e = (u, v) \), \( \ell(e) = \ell(u, v) \) is its length.

- Given nodes \( s, t \) find shortest path from \( s \) to \( t \).
- Given node \( s \) find shortest path from \( s \) to all other nodes.

Restrict attention to directed graphs

Undirected graph problem can be reduced to directed graph problem - how?

- Given undirected graph \( G \), create a new directed graph \( G' \) by replacing each edge \( \{u, v\} \) in \( G \) by \( (u, v) \) and \( (v, u) \) in \( G' \).
- Set \( \ell(u, v) = \ell(v, u) = \ell(\{u, v\}) \).
Single-Source Shortest Paths: Non-Negative Edge Lengths

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Single-Source Shortest Paths: Non-Negative Edge Lengths

**Single-Source Shortest Path Problems**

- **Input** A (undirected or directed) graph $G = (V, E)$ with non-negative edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

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- Restrict attention to directed graphs
- Undirected graph problem can be reduced to directed graph problem - how?
  - Given undirected graph $G$, create a new directed graph $G'$ by replacing each edge $\{u, v\}$ in $G$ by $(u, v)$ and $(v, u)$ in $G'$.
  - Set $\ell(u, v) = \ell(v, u) = \ell(\{u, v\})$.

Exercise: show reduction works
**Special case:** All edge lengths are 1.
- Run \( \text{BFS}(s) \) to get shortest path distances from \( s \) to all other nodes.
- \( O(m + n) \) time algorithm.

**Special case:** Suppose \( \ell(e) \) is an integer for all \( e \)? Can we use \( \text{BFS} \)? Reduce to unit edge-length problem by placing \( \ell(e) - 1 \) dummy nodes on \( e \)

Let \( L = \max_e \ell(e) \). New graph has \( O(mL) \) edges and \( O(mL + n) \) nodes. \( \text{BFS} \) takes \( O(mL + n) \) time. Not efficient if \( L \) is large.
Special case: All edge lengths are 1.
- Run $\text{BFS}(s)$ to get shortest path distances from $s$ to all other nodes.
- $O(m + n)$ time algorithm.

Special case: Suppose $\ell(e)$ is an integer for all $e$?
Can we use $\text{BFS}$? Reduce to unit edge-length problem by placing $\ell(e) - 1$ dummy nodes on $e$

Let $L = \max_e \ell(e)$. New graph has $O(mL)$ edges and $O(mL + n)$ nodes. $\text{BFS}$ takes $O(mL + n)$ time. Not efficient if $L$ is large.
Single-Source Shortest Paths via BFS

Special case: All edge lengths are 1.
- Run **BFS** \((s)\) to get shortest path distances from \(s\) to all other nodes.
- \(O(m + n)\) time algorithm.

Special case: Suppose \(\ell(e)\) is an integer for all \(e\)?
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Let \( L = \max_e \ell(e) \). New graph has \( O(mL) \) edges and \( O(mL + n) \) nodes. BFS takes \( O(mL + n) \) time. Not efficient if \( L \) is large.
Towards an algorithm

Why does **BFS** work?

**BFS**($s$) explores nodes in increasing distance from $s$

**Lemma**

Let $G$ be a directed graph with non-negative edge lengths. Let $\text{dist}(s, v)$ denote the shortest path length from $s$ to $v$. If $s = v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is a shortest path from $s$ to $v_k$, then for $1 \leq i < k$:

- $s = v_0 \to v_1 \to v_2 \to \ldots \to v_i$ is a shortest path from $s$ to $v_i$
- $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$.

**Proof.**

Suppose not. Then for some $i < k$ there is a path $P'$ from $s$ to $v_i$ of length strictly less than that of $s = v_0 \to v_1 \to \ldots \to v_i$. Then $P'$ concatenated with $v_i \to v_{i+1} \ldots \to v_k$ contains a strictly shorter
Towards an algorithm

Why does **BFS** work?

**BFS**\( (s) \) explores nodes in increasing distance from \( s \)

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**Lemma**

Let \( G \) be a directed graph with non-negative edge lengths. Let \( \text{dist}(s, v) \) denote the shortest path length from \( s \) to \( v \). If \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \) is a shortest path from \( s \) to \( v_k \) then for \( 1 \leq i < k \):

- \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i \) is a shortest path from \( s \) to \( v_i \)
- \( \text{dist}(s, v_i) \leq \text{dist}(s, v_k) \).

**Proof.**

Suppose not. Then for some \( i < k \) there is a path \( P' \) from \( s \) to \( v_i \) of length strictly less than that of \( s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i \). Then \( P' \) concatenated with \( v_i \rightarrow v_{i+1} \cdots \rightarrow v_k \) contains a strictly shorter
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Why does **BFS** work?

**BFS**(*s*) explores nodes in increasing distance from *s*

**Lemma**

Let **G** be a directed graph with non-negative edge lengths. Let 
**dist**(*s*, *v*) denote the shortest path length from *s* to *v*. If 
*s* = *v*₀ → *v*₁ → *v*₂ → ... → *v*ₖ is a shortest path from *s* to *v*ₖ 
then for 1 ≤ *i* < *k*:

- *s* = *v*₀ → *v*₁ → *v*₂ → ... → *v*ᵢ is a shortest path from *s* to *v*ᵢ
- **dist**(*s*, *v*ᵢ) ≤ **dist**(*s*, *v*ₖ).

**Proof.**

Suppose not. Then for some *i* < *k* there is a path **P**′ from *s* to *v*ᵢ of 
length strictly less than that of *s* = *v*₀ → *v*₁ → ... → *v*ᵢ. Then **P**′ 
concatenated with *v*ᵢ → *v*ᵢ₊₁ ... → *v*ₖ contains a strictly shorter
Towards an algorithm

Lemma

Let $G$ be a directed graph with non-negative edge lengths. Let $\text{dist}(s, v)$ denote the shortest path length from $s$ to $v$. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

- $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
- $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$.

Proof.

Suppose not. Then for some $i < k$ there is a path $P'$ from $s$ to $v_i$ of length strictly less than that of $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$. Then $P'$ concatenated with $v_i \rightarrow v_{i+1} \ldots \rightarrow v_k$ contains a strictly shorter path to $v_k$ than $s = v_0 \rightarrow v_1 \ldots \rightarrow v_k$. \qed
A proof by picture

$s = v_0$

$\text{Shortest path from } v_0 \text{ to } v_6$
A proof by picture

Shorter path from \( v_0 \) to \( v_4 \)

\[ s = v_0 \]

Shortest path from \( v_0 \) to \( v_6 \)
A proof by picture

\[ s = v_0 \]

A shorter path from \( v_0 \) to \( v_6 \). A contradiction.

Shortest path from \( v_0 \) to \( v_6 \)
A Basic Strategy

Explore vertices in increasing order of distance from $s$:
(For simplicity assume that nodes are at different distances from $s$
and that no edge has zero length)

Initialize for each node $v$, $\text{dist}(s, v) = \infty$
Initialize $S = \emptyset$,
for $i = 1$ to $|V|$ do
  (* Invariant: $S$ contains the $i-1$ closest nodes to $s$ *)
  Among nodes in $V \setminus S$, find the node $v$ that is the
  $i$th closest to $s$
  Update $\text{dist}(s, v)$
  $S = S \cup \{v\}$

How can we implement the step in the for loop?
A Basic Strategy

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How can we implement the step in the for loop?
Finding the $i$th closest node

- $S$ contains the $i - 1$ closest nodes to $s$
- Want to find the $i$th closest node from $V - S$.

What do we know about the $i$th closest node?

**Claim**

Let $P$ be a shortest path from $s$ to $v$ where $v$ is the $i$th closest node. Then, all intermediate nodes in $P$ belong to $S$.

**Proof.**

If $P$ had an intermediate node $u$ not in $S$ then $u$ will be closer to $s$ than $v$. Implies $v$ is not the $i$th closest node to $s$ - recall that $S$ already has the $i - 1$ closest nodes.
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Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example

```
0 9 6 6 25 6 9 0
a

b 9

6

c

13 8 20

d

10 18 30

11

e

6

19

6

f 6

16 25

g

11

h
```
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Finding the $i$th closest node repeatedly

An example
Corollary

*The *ith closest node is adjacent to $S$.  

Finding the *ith closest node
Finding the \( i \)th closest node

- \( S \) contains the \( i - 1 \) closest nodes to \( s \)
- Want to find the \( i \)th closest node from \( V - S \).

For each \( u \in V - S \) let \( P(s, u, S) \) be a shortest path from \( s \) to \( u \) using only nodes in \( S \) as intermediate vertices.
- Let \( d'(s, u) \) be the length of \( P(s, u, S) \)

Observations: for each \( u \in V - S \),
- \( \text{dist}(s, u) \leq d'(s, u) \) since we are constraining the paths
- \( d'(s, u) = \min_{a \in S}(\text{dist}(s, a) + \ell(a, u)) \) - Why?

**Lemma**

If \( v \) is the \( i \)th closest node to \( s \), then \( d'(s, v) = \text{dist}(s, v) \).
Finding the $i$th closest node

- $S$ contains the $i-1$ closest nodes to $s$
- Want to find the $i$th closest node from $V - S$.
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Finding the $i$th closest node

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**Lemma**

*If $v$ is the $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.***
Finding the $i$th closest node

**Lemma**

If $v$ is an $i$th closest node to $s$, then $d'(s, v) = \text{dist}(s, v)$.

**Proof.**

Let $v$ be the $i$th closest node to $s$. Then there is a shortest path $P$ from $s$ to $v$ that contains only nodes in $S$ as intermediate nodes (see previous claim). Therefore $d'(s, v) = \text{dist}(s, v)$. \qed
Finding the \textbf{i}th closest node

\textbf{Lemma}

If \( v \) is an \textbf{i}th closest node to \( s \), then \( d'(s, v) = \text{dist}(s, v) \).

\textbf{Corollary}

The \textbf{i}th closest node to \( s \) is the node \( v \in V - S \) such that
\[ d'(s, v) = \min_{u \in V - S} d'(s, u). \]

\textbf{Proof.}

For every node \( u \in V - S \), \( \text{dist}(s, u) \leq d'(s, u) \) and for the \textbf{i}th closest node \( v \), \( \text{dist}(s, v) = d'(s, v) \). Moreover, \( \text{dist}(s, u) \geq \text{dist}(s, v) \) for each \( u \in V - S \).
Algorithm

Initialize for each node $v$: $\text{dist}(s, v) = \infty$

Initialize $S = \emptyset$, $d'(s, s) = 0$

for $i = 1$ to $|V|$ do

(* Invariant: $S$ contains the $i-1$ closest nodes to $s$ *)

(* Invariant: $d'(s, u)$ is shortest path distance from $u$ to $s$ using only $S$ as intermediate nodes*)

Let $v$ be such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$

$\text{dist}(s, v) = d'(s, v)$

$S = S \cup \{v\}$

for each node $u$ in $V \setminus S$

compute $d'(s, u) = \min_{a \in S} \left( \text{dist}(s, a) + \ell(a, u) \right)$

Correctness: By induction on $i$ using previous lemmas.

Running time: $O(n \cdot (n + m))$ time.

- $n$ outer iterations. In each iteration, $d'(s, u)$ for each $u$ by scanning all edges out of nodes in $S$; $O(m + n)$ time/iteration.
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Example
Example
Example

![Graph Diagram]

- **Example Network**: A directed graph with labeled edges showing the weights.
- **Vertices**: `s`, `2`, `3`, `6`, `4`, `5`, `7`, `t`, `0`, `9`, `19`, `36`, `25`, `38`, `13`.
- **Edges & Weights**: Various weighted connections between nodes, e.g., `s` to `2` with weight 9, `2` to `3` with weight 10, etc.

This example illustrates a network problem, possibly for algorithm demonstration or analysis.
Example
Example
Example
Improved Algorithm

- Main work is to compute the $d'(s, u)$ values in each iteration.
- $d'(s, u)$ changes from iteration $i$ to $i + 1$ only because of the node $v$ that is added to $S$ in iteration $i$.

Initialize for each node $v$, $\text{dist}(s, v) = d'(s, v) = \infty$
Initialize $S = \emptyset$, $d'(s,s) = 0$

for $i = 1$ to $|V|$ do
  
  // $S$ contains the $i-1$ closest nodes to $s$,
  // and the values of $d'(s,u)$ are current
  Let $v$ be such that $d'(s,v) = \min_{u \in V-S} d'(s,u)$
  $\text{dist}(s, v) = d'(s, v)$
  $S = S \cup \{v\}$
  Update $d'(s,u)$ for each $u$ in $V-S$ as follows:
  \[ d'(s, u) = \min(d'(s, u), \text{dist}(s, v) + \ell(v, u)) \]

Running time: $O(m + n^2)$ time.

- $n$ outer iterations and in each iteration following steps
- updating $d'(s, u)$ after $v$ added takes $O(\text{deg}(v))$, time so total
  running time is $O(m + n^2)$.
Improved Algorithm

- Main work is to compute the $d'(s, u)$ values in each iteration
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  work is $O(m)$ since a node enters $S$ only once

Finding $v$ from $d'(s, u)$ values is $O(n)$ time

Sariel (UIUC) CS473 Spring 2011
Improved Algorithm

Initialize for each node $v$, $\text{dist}(s, v) = d'(s, v) = \infty$
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for $i = 1$ to $|V|$ do

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Let $v$ be such that $d'(s, v) = \min_{u \in V - S} d'(s, u)$
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- Finding $v$ from $d'(s, u)$ values is $O(n)$ time
Dijkstra’s Algorithm

- eliminate \( d'(s, u) \) and let \( \text{dist}(s, u) \) maintain it
- update \( \text{dist} \) values after adding \( v \) by scanning edges out of \( v \)

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  \( S = S \cup \{v\} \)
  for each \( u \) in \( \text{Adj}(v) \) do
    \( \text{dist}(s, u) = \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u)) \)

Priority Queues to maintain \( \text{dist} \) values for faster running time

- Using heaps and standard priority queues: \( O((m + n) \log n) \)
- Using Fibonacci heaps: \( O(m + n \log n) \).
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Priority Queues to maintain $\text{dist}$ values for faster running time

- Using heaps and standard priority queues: $O((m + n) \log n)$
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Priority Queues

Data structure to store a set \( S \) of \( n \) elements where each element \( v \in S \) has an associated real/integer key \( k(v) \) such that the following operations

- **makeQ**: create an empty queue
- **findMin**: find the minimum key in \( S \)
- **extractMin**: Remove \( v \in S \) with smallest key and return it
- **add(v, k(v))**: Add new element \( v \) with key \( k(v) \) to \( S \)
- **delete(v)**: Remove element \( v \) from \( S \)
- **decreaseKey(v, k'(v))**: decrease key of \( v \) from \( k(v) \) (current key) to \( k'(v) \) (new key). Assumption: \( k'(v) \leq k(v) \)
- **meld**: merge two separate priority queues into one can be performed in \( O(\log n) \) time each.

decreaseKey via delete and add
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- **meld**: merge two separate priority queues into one

Can be performed in $O(\log n)$ time each.

decreaseKey via delete and add
Dijkstra’s Algorithm using Priority Queues

\[ Q = \text{makePQ()} \]
\[ \text{insert}(Q, (s, 0)) \]
\[ \text{for each node } u \neq s \text{ do} \]
\[ \quad \text{insert}(Q, (u, \infty)) \]
\[ S = \emptyset \]
\[ \text{for } i = 1 \text{ to } |V| \text{ do} \]
\[ \quad (v, \text{dist}(s, v)) = \text{extractMin}(Q) \]
\[ \quad S = S \cup \{v\} \]
\[ \quad \text{For each } u \text{ in } \text{Adj}(v) \text{ do} \]
\[ \quad \quad \text{decreaseKey}(Q, (u, \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u)))) \]

Priority Queue operations:

- \( O(n) \) insert operations
- \( O(n) \) extractMin operations
- \( O(m) \) decreaseKey operations
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

- All operations can be done in $O(\log n)$ time

Dijkstra's algorithm can be implemented in $O((n + m) \log n)$ time.
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

- All operations can be done in $O(\log n)$ time

Dijkstra’s algorithm can be implemented in $O((n + m) \log n)$ time.
Fibonacci Heaps

- `extractMin`, `add`, `delete`, `meld` in $O(\log n)$ time
- `decreaseKey` in $O(1)$ amortized time: $\ell$ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
- Relaxed Heaps: `decreaseKey` in $O(1)$ worst case time but at the expense of `meld` (not necessary for Dijkstra’s algorithm)

---

Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time. If $m = \Omega(n \log n)$, running time is linear in input size.

---

Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
Fibonacci Heaps

- extractMin, add, delete, meld in $O(\log n)$ time
- decreaseKey in $O(1)$ amortized time: $\ell$ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
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## Fibonacci Heaps

- `extractMin`, `add`, `delete`, `meld` in $\mathcal{O}(\log n)$ time
- `decreaseKey` in $\mathcal{O}(1)$ amortized time: $\ell$ `decreaseKey` operations for $\ell \geq n$ take together $\mathcal{O}(\ell)$ time
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Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
Dijkstra’s algorithm finds the shortest path distances from s to V.

**Question:** How do we find the paths themselves?

```
Q = makePQ()
insert(Q, (s, 0))
prev(s) = null
for each node u ≠ s do
    insert(Q, (u, ∞))
    prev(u) = null

S = ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s, v) + ℓ(v, u) < dist(s, u)) then
            decreaseKey(Q, (u, dist(s, v) + ℓ(v, u)))
            prev(u) = v
```
Dijkstra’s algorithm finds the shortest path distances from s to V.

**Question:** How do we find the paths themselves?

```plaintext
Q = makePQ()
insert(Q, (s, 0))
prev(s) = null
for each node u ≠ s do
    insert(Q, (u, ∞))
    prev(u) = null

S = ∅
for i = 1 to |V| do
    (v, dist(s, v)) = extractMin(Q)
    S = S ∪ {v}
    for each u in Adj(v) do
        if (dist(s, v) + ℓ(v, u) < dist(s, u)) then
            decreaseKey(Q, (u, dist(s, v) + ℓ(v, u))
            prev(u) = v
```
Lemma

*The edge set* \( (u, \text{prev}(u)) \) *is the reverse of a shortest path tree rooted at* \( s \). *For each* \( u \), *the reverse of the path from* \( u \) *to* \( s \) *in the tree is a shortest path from* \( s \) *to* \( u \).*

Proof Sketch.

- The edgeset \( \{(u, \text{prev}(u)) \mid u \in V\} \) induces a directed in-tree rooted at \( s \) (Why?)
- Use induction on \(|S|\) to argue that the tree is a shortest path tree for nodes in \( V \).
Shortest paths to s

Dijkstra’s algorithm gives shortest paths from s to all nodes in V.

How do we find shortest paths from all of V to s?

- In undirected graphs shortest path from s to u is a shortest path from u to s so there is no need to distinguish.
- In directed graphs, use Dijkstra’s algorithm in $G^{rev}$!
Shortest paths to $s$

Dijkstra’s algorithm gives shortest paths from $s$ to all nodes in $V$.

How do we find shortest paths from all of $V$ to $s$?

- In undirected graphs shortest path from $s$ to $u$ is a shortest path from $u$ to $s$ so there is no need to distinguish.
- In directed graphs, use Dijkstra’s algorithm in $G_{rev}$!