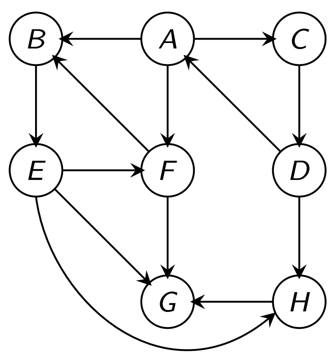
DFS in Directed Graphs, Strong Connected Components, and DAGs

Lecture 2

January 20, 2011

Strong Connected Components (SCCs)



Algorithmic Problem

Find all SCCs of a given directed graph.

Previous lecture: saw an $O(n \cdot (n + m))$ time algorithm.
This lecture: $O(n + m)$ time algorithm.

Graph of SCCs

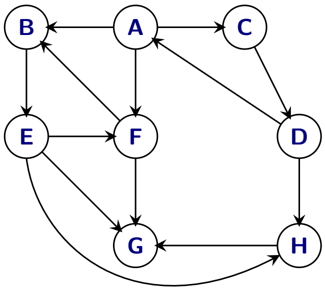


Figure: Graph **G**

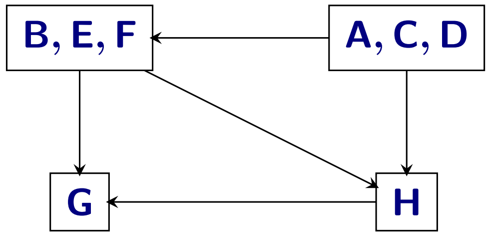


Figure: Graph of s **G**

Meta-graph of SCCs

Let $\mathbf{S_1, S_2, \ldots S_k}$ be the SCCs of $\mathbf{G}$. The graph of SCCs is $\mathbf{G}^{\mathrm{SCC}}$

- Vertices are $\mathbf{S_1, S_2, \ldots S_k}$
- There is an edge $\mathbf{(S_i, S_j)}$ if there is some $\mathbf{u} \in \mathbf{S_i}$ and $\mathbf{v} \in \mathbf{S_j}$ such that $\mathbf{(u, v)}$ is an edge in $\mathbf{G}$.

Reversal and SCCs

Proposition

For any graph $\mathbf{G}$, the graph of SCCs of $\mathbf{G}^{\mathrm{rev}}$ is the same as the reversal of $\mathbf{G}^{\mathrm{SCC}}$.

Proof.

Exercise. □

SCCs and DAGs

Proposition

For any graph $\mathbf{G}$, the graph $\mathbf{G}^{\mathrm{SCC}}$ has no directed cycle.

Proof.

If $\mathbf{G}^{\mathrm{SCC}}$ has a cycle $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_k$ then $\mathbf{S}_1 \cup \mathbf{S}_2 \cup \cdots \cup \mathbf{S}_k$ is an SCC in $\mathbf{G}$. Formal details: exercise. $\square$

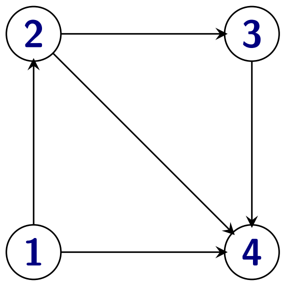
Part I

Directed Acyclic Graphs

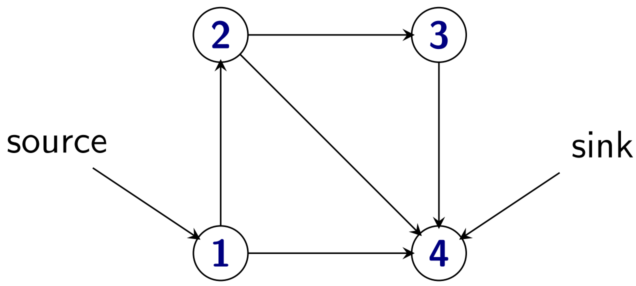
Directed Acyclic Graphs

Definition

A directed graph **G** is a **directed acyclic graph** (DAG) if there is no directed cycle in **G**.



Sources and Sinks



Definition

- A vertex **u** is a **source** if it has no in-coming edges.
- A vertex **u** is a **sink** if it has no out-going edges.

Simple DAG Properties

- Every DAG $\mathbf{G}$ has at least one source and at least one sink.
- If $\mathbf{G}$ is a DAG if and only if $\mathbf{G}^{\mathsf{rev}}$ is a DAG.
- $\mathbf{G}$ is a DAG if and only each node is in its own strong connected component.

Formal proofs: exercise.

Topological Ordering/Sorting

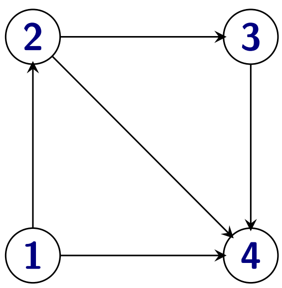


Figure: Graph $\mathbf{G}$

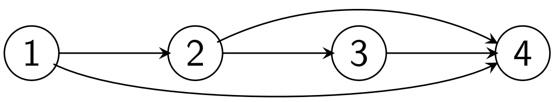


Figure: Topological Ordering of $\mathbf{G}$

Definition

A **topological ordering**/**topological sorting** of $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ is an ordering $<$ on $\mathbf{V}$ such that if $(\mathbf{u}, \mathbf{v}) \in \mathbf{E}$ then $\mathbf{u} < \mathbf{v}$.

DAGs and Topological Sort

Lemma

A directed graph $\mathbf{G}$ can be topologically ordered iff it is a DAG.

Proof.

Only if: Suppose $\mathbf{G}$ is not a DAG and has a topological ordering $<$.
$\mathbf{G}$ has a cycle $\mathbf{C} = \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_1$.
Then $\mathbf{u}_1 < \mathbf{u}_2 < \ldots < \mathbf{u}_k < \mathbf{u}_1$! A contradiction. □

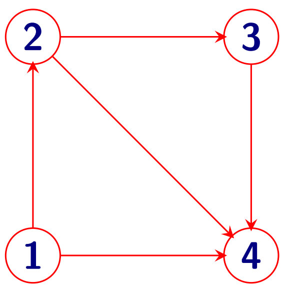
Proof.

If: Consider the following algorithm:

- Pick a source $\mathbf{u}$, output it.
- Remove $\mathbf{u}$ and all edges out of $\mathbf{u}$.
- Repeat until graph is empty.
- Exercise: prove this gives an ordering.

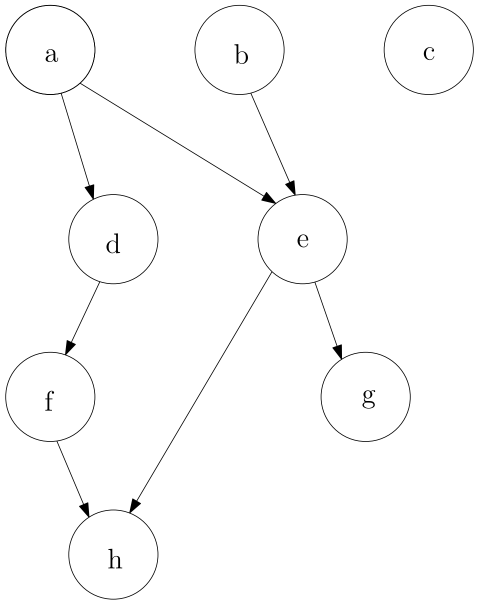
□

Topological Sort: An Example



Output: 1 2 3 4

Topological Sort: Another Example



DAGs and Topological Sort

Note: A DAG $\mathbf{G}$ may have many different topological sorts.

Question: What is a DAG with the most number of distinct topological sorts for a given number $\mathbf{n}$ of vertices?

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Using DFS...

... to check for Acylicity and compute Topological Ordering

Question

Given **G**, is it a DAG? If it is, generate a topological sort.

DFS based algorithm:

- Compute **DFS(G)**
- If there is a back edge then **G** is not a DAG.
- Otherwise output nodes in decreasing post-visit order.

Correctness relies on the following:

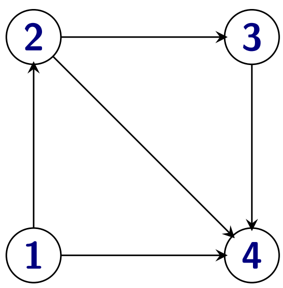
Proposition

G *is a* DAG *iff there is no back-edge in* **DFS(G)**.

Proposition

If **G** *is a* DAG *and* $post(v) > post(u)$, *then* **(u, v)** *is not in* **G**.

Example



Back edge and Cycles

Proposition

$\mathbf{G}$ *has a cycle iff there is a back-edge in* $\mathbf{DFS(G)}$.

Proof.

If: (u, v) is a back edge implies there is a cycle C consisting of the path from v to u in **DFS** search tree and the edge (u, v).

Only if: Suppose there is a cycle $C = v_1 \to v_2 \to \ldots \to v_k \to v_1$.
Let v_i be first node in C visited in **DFS**.
All other nodes in C are descendents of v_i since they are reachable from v_i.
Therefore, (v_{i-1}, v_i) (or (v_k, v_1) if $i = 1$) is a back edge. $\square$

DAGs and Partial Orders

Definition

A **partially ordered set** is a set S along with a binary relation $\preceq$ such that $\preceq$ is

1. reflexive ($a \preceq a$ for all $a \in V$),
2. anti-symmetric ($a \preceq b$ and $a \neq b$ implies $b \not\preceq a$), and
3. transitive ($a \preceq b$ and $b \preceq c$ implies $a \preceq c$).

Example: For numbers in the plane define $(x, y) \preceq (x', y')$ iff $x \leq x'$ and $y \leq y'$.
Observation: A *finite* partially ordered set is equivalent to a DAG.

Observation: A topological sort of a DAG corresponds to a complete (or total) ordering of the underlying partial order.

Part II

Linear time algorithm for finding all strong connected components of a directed graph

Finding all SCCs of a Directed Graph

Problem

Given a directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, output *all* its strong connected components.

Straightforward algorithm:

For each vertex $\mathbf{u} \in \mathbf{V}$ do
    find $\mathbf{SCC}(\mathbf{G}, \mathbf{u})$ the strong component containing $\mathbf{u}$ as follows:
        Obtain $\mathtt{rch}(\mathbf{G}, \mathbf{u})$ using $\mathbf{DFS}(\mathbf{G}, \mathbf{u})$
        Obtain $\mathtt{rch}(\mathbf{G}^{\mathrm{rev}}, \mathbf{u})$ using $\mathbf{DFS}(\mathbf{G}^{\mathrm{rev}}, \mathbf{u})$
        Output $\mathbf{SCC}(\mathbf{G}, \mathbf{u}) = \mathtt{rch}(\mathbf{G}, \mathbf{u}) \cap \mathtt{rch}(\mathbf{G}^{\mathrm{rev}}, \mathbf{u})$

Running time: $\mathbf{O}(\mathbf{n}(\mathbf{n} + \mathbf{m}))$

Is there an $\mathbf{O}(\mathbf{n} + \mathbf{m})$ time algorithm?

Structure of a Directed Graph

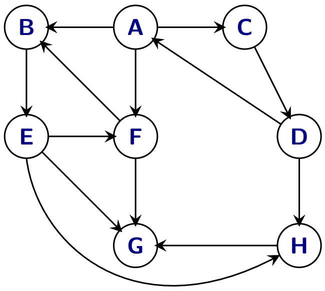


Figure: Graph $\mathbf{G}$

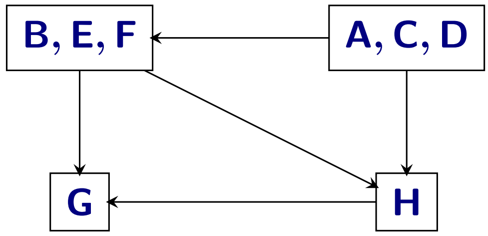


Figure: Graph of SCCs $\mathbf{G}^{\mathrm{SCC}}$

Proposition

For a directed graph $\mathbf{G}$*, its meta-graph* $\mathbf{G}^{\mathrm{SCC}}$ *is a* DAG*.*

Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph.

Algorithm

- Let $\mathbf{u}$ be a vertex in a sink SCC of $\mathbf{G}^{\mathrm{SCC}}$
- Do **DFS**$(\mathbf{u})$ to compute $\mathrm{SCC}(\mathbf{u})$
- Remove $\mathrm{SCC}(\mathbf{u})$ and repeat

Justification

- **DFS**$(\mathbf{u})$ only visits vertices (and edges) in $\mathrm{SCC}(\mathbf{u})$
- **DFS**$(\mathbf{u})$ takes time proportional to size of $\mathrm{SCC}(\mathbf{u})$
- Therefore, total time $\mathbf{O}(\mathbf{n}+\mathbf{m})$!

Big Challenge(s)

How do we find a vertex in the sink SCC of $\mathbf{G}^{\mathrm{SCC}}$?

Can we obtain an *implicit* topological sort of $\mathbf{G}^{\mathrm{SCC}}$ without computing $\mathbf{G}^{\mathrm{SCC}}$?

Answer: **DFS(G)** gives some information!

Post-visit times of SCCs

Definition

Given $\mathbf{G}$ and a SCC $\mathbf{S}$ of $\mathbf{G}$, define $\mathrm{post}(\mathbf{S}) = \max_{\mathbf{u} \in \mathbf{S}} \mathrm{post}(\mathbf{u})$ where post numbers are with respect to some **DFS(G)**.

An Example

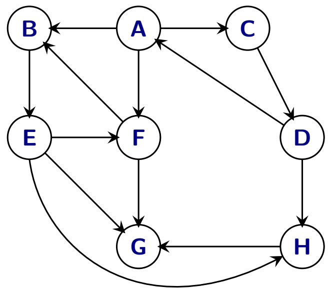


Figure: Graph **G**

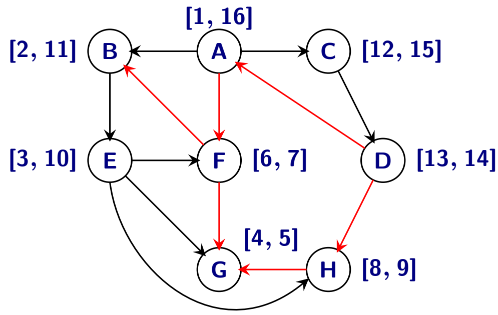


Figure: Graph with pre-post times for **DFS**(**A**); black edges in tree

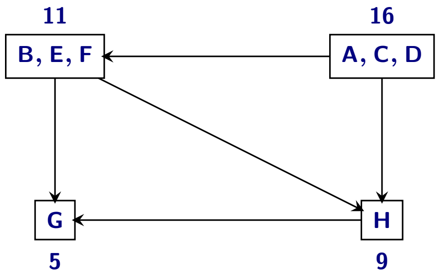


Figure: $\mathbf{G}^{\mathrm{SCC}}$ with post times

$\mathbf{G}^{\mathrm{SCC}}$ and post-visit times

Proposition

If $\mathbf{S}$ *and* $\mathbf{S}'$ *are* SCC*s in* $\mathbf{G}$ *and* $(\mathbf{S}, \mathbf{S}')$ *is an edge in* $\mathbf{G}^{\mathrm{SCC}}$ *then* $\mathrm{post}(\mathbf{S}) > \mathrm{post}(\mathbf{S}')$.

Proof.

Let $\mathbf{u}$ be first vertex in $\mathbf{S} \cup \mathbf{S}'$ that is visited.

- If $\mathbf{u} \in \mathbf{S}$ then all of $\mathbf{S}'$ will be explored before **DFS**$(\mathbf{u})$ completes.
- If $\mathbf{u} \in \mathbf{S}'$ then all of $\mathbf{S}'$ will be explored before any of $\mathbf{S}$.

$\square$

A False Statement: If $\mathbf{S}$ and $\mathbf{S}'$ are SCCs in $\mathbf{G}$ and $(\mathbf{S}, \mathbf{S}')$ is an edge in $\mathbf{G}^{\mathrm{SCC}}$ then for *every* $\mathbf{u} \in \mathbf{S}$ and $\mathbf{u}' \in \mathbf{S}'$, $\mathrm{post}(\mathbf{u}) > \mathrm{post}(\mathbf{u}')$.

Topological ordering of G^{SCC}

Corollary

Ordering SCCs in decreasing order of $\mathrm{post}(\mathbf{S})$ gives a topological ordering of G^{SCC}

Recall: for a DAG, ordering nodes in decreasing post-visit order gives a topological sort.

So...
DFS(G) gives some information on topological ordering of G^{SCC}!

Finding Sources

Proposition

The vertex $\mathbf{u}$ with the highest post visit time belongs to a source SCC in G^{SCC}

Proof.

- $\mathrm{post}(\mathrm{SCC}(\mathbf{u})) = \mathrm{post}(\mathbf{u})$
- Thus, $\mathrm{post}(\mathrm{SCC}(\mathbf{u}))$ is highest and will be output first in topological ordering of G^{SCC}.

□

Finding Sinks

Proposition

The vertex $\mathbf{u}$ *with highest post visit time in* $\mathbf{DFS}(\mathbf{G}^{\mathrm{rev}})$ *belongs to a sink SCC of* $\mathbf{G}$.

Proof.

- $\mathbf{u}$ belongs to source SCC of $\mathbf{G}^{\mathrm{rev}}$
- Since graph of SCCs of $\mathbf{G}^{\mathrm{rev}}$ is the reverse of $\mathbf{G}^{\mathrm{SCC}}$, $\mathrm{SCC}(\mathbf{u})$ is sink SCC of $\mathbf{G}$. □

Linear Time Algorithm

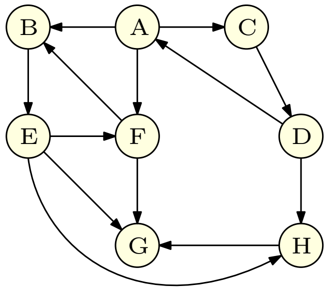
```
Do DFS(G^rev) and sort vertices in decreasing post order.
Mark all nodes as unvisited
for each u in the computed order do
    if u is not visited then
        DFS(u)
        Let S_u be the nodes reached by u
        Output S_u as a strong connected component
        Remove S_u from G
```

Analysis

Running time is $\mathbf{O}(\mathbf{n}+\mathbf{m})$. (Exercise)

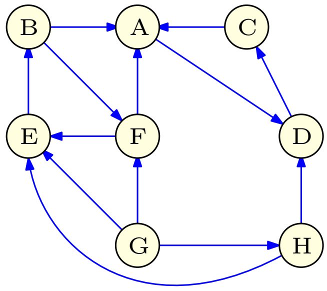
Linear Time Algorithm: An Example - Initial steps

Graph **G**:



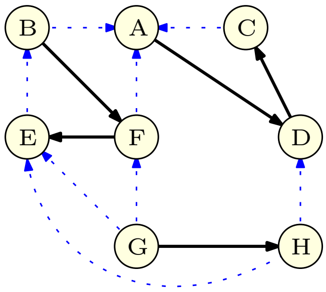
$\Longrightarrow$

Reverse graph $\mathbf{G}^{\mathbf{rev}}$:



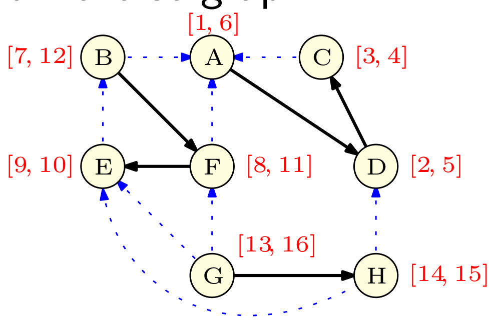
DFS of reverse graph:

$\Longrightarrow$



$\Longrightarrow$

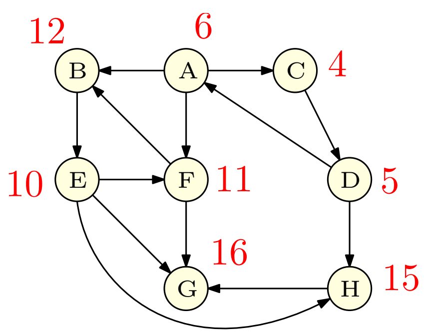
Pre/Post **DFS** numbering of reverse graph:



Linear Time Algorithm: An Example

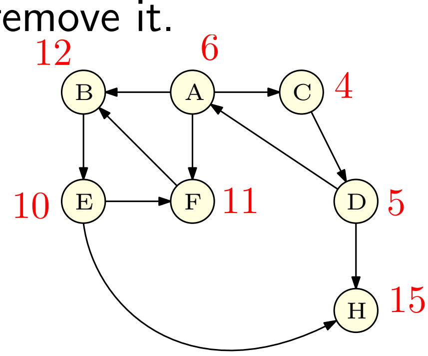
Removing connected components: 1

Original graph **G** with rev post numbers:



$\Longrightarrow$

Do **DFS** from vertex **G** remove it.

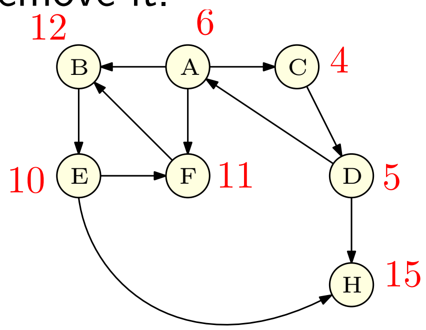


SCC computed:
{G}

Linear Time Algorithm: An Example

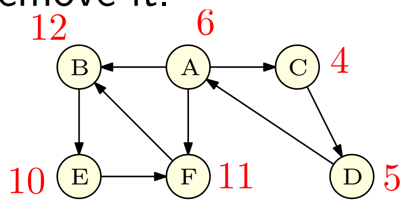
Removing connected components: 2

Do **DFS** from vertex **G** remove it.



$\Longrightarrow$

Do **DFS** from vertex **H**, remove it.



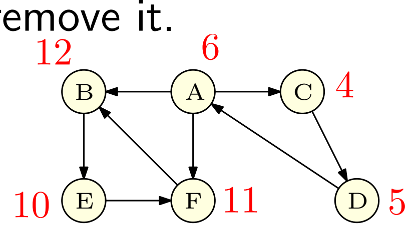
SCC computed:
{G}

SCC computed:
{G}, {H}

Linear Time Algorithm: An Example

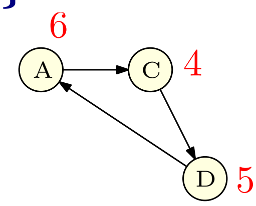
Removing connected components: 3

Do **DFS** from vertex **H**, remove it.



$\Longrightarrow$

Do **DFS** from vertex **F**
Remove visited vertices:
{F, B, E}.



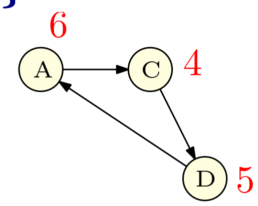
SCC computed:
{G}, {H}

SCC computed:
{G}, {H}, {F, B, E}

Linear Time Algorithm: An Example

Removing connected components: 4

Do **DFS** from vertex **F**
Remove visited vertices:
$\{F, B, E\}$.



$\Longrightarrow$

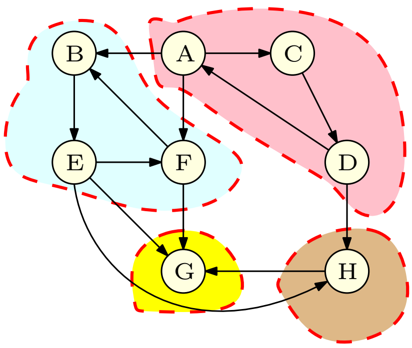
Do **DFS** from vertex **A**
Remove visited vertices:
$\{A, C, D\}$.

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}$

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$

Linear Time Algorithm: An Example

Final result



SCC computed:
$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$
Which is the correct answer!

Obtaining the meta-graph from strong connected components

Exercise: Given all the strong connected components of a directed graph $G = (V, E)$ show that the meta-graph G^{SCC} can be obtained in $O(m + n)$ time.

Correctness: more details

- let $S_1, S_2, \ldots, S_k$ be strong components in G
- Strong components of G^{rev} and G are same and meta-graph of G is reverse of meta-graph of G^{rev}.
- consider $DFS(G^{rev})$ and let $u_1, u_2, \ldots, u_k$ be such that $\mathrm{post}(u_i) = \mathrm{post}(S_i) = \max_{v \in S_i} \mathrm{post}(v)$.
- Assume without loss of generality that $post(u_k) > post(u_{k-1}) \geq \ldots \geq post(u_1)$ (renumber otherwise). Then $S_k, S_{k-1}, \ldots, S_1$ is a topological sort of meta-graph of G^{rev} and hence $S_1, S_2, \ldots, S_k$ is a topological sort of the meta-graph of G.
- u_k has highest post number and $DFS(u_k)$ will explore all of S_k which is a sink component in G.
- After S_k is removed u_{k-1} has highest post number and $DFS(u_{k-1})$ will explore all of S_{k-1} which is a sink component in remaining graph $G - S_k$. Formal proof by induction.

Part III

An Application to make

make Utility [Feldman]

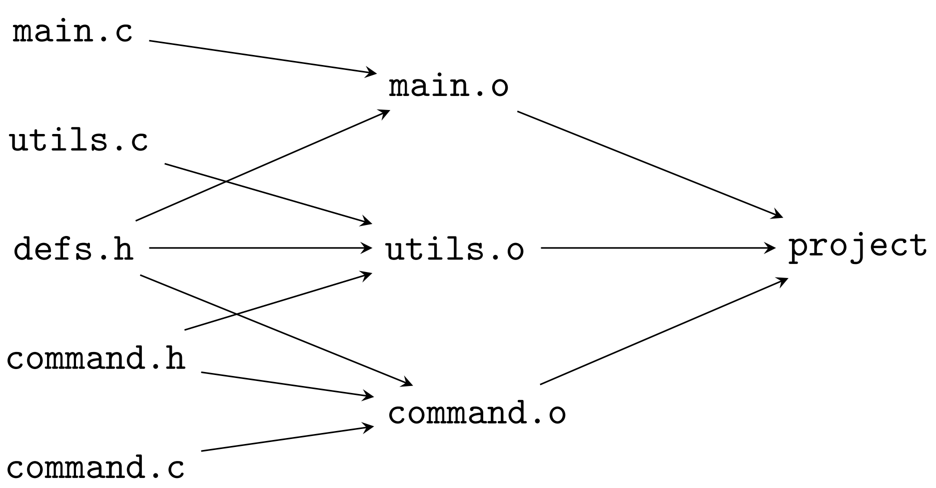
- Unix utility for automatically building large software applications
- A makefile specifies
  - Object files to be created,
  - Source/object files to be used in creation, and
  - How to create them

An Example makefile

```
project:  main.o utils.o command.o
    cc -o project main.o utils.o command.o

main.o:  main.c defs.h
    cc -c main.c
utils.o:  utils.c defs.h command.h
    cc -c utils.c
command.o:  command.c defs.h command.h
    cc -c command.c
```


makefile as a Digraph



Computational Problems for make

- Is the makefile reasonable?
- If it is reasonable, in what order should the object files be created?
- If it is not reasonable, provide helpful debugging information.
- If some file is modified, find the fewest compilations needed to make application consistent.

Algorithms for make

- Is the makefile reasonable? Is **G** a DAG?
- If it is reasonable, in what order should the object files be created? Find a topological sort of a DAG.
- If it is not reasonable, provide helpful debugging information. Output a cycle. More generally, output all strong connected components.
- If some file is modified, find the fewest compilations needed to make application consistent.
  - Find all vertices reachable (using **DFS**/**BFS**) from modified files in directed graph, and recompile them in proper order. Verify that one can find the files to recompile and the ordering in linear time.

Take away Points

- Given a directed graph **G**, its SCCs and the associated acyclic meta-graph $\mathbf{G}^{\mathrm{SCC}}$ give a structural decomposition of **G** that should be kept in mind.
- There is a **DFS** based linear time algorithm to compute all the SCCs and the meta-graph. Properties of **DFS** crucial for the algorithm.
- DAGs arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).