CS 473: Undergraduate Algorithms, Spring 2010
Homework 0

Due Tuesday, January 26, 2009 **in class**

- This homework tests your familiarity with prerequisite material—big-Oh notation, elementary algorithms and data structures, recurrences, graphs, and most importantly, induction—to help you identify gaps in your background knowledge. **You are responsible for filling those gaps.** The early chapters of any algorithms textbook should be sufficient review, but you may also want consult your favorite discrete mathematics and data structures textbooks. If you need help, please ask in office hours and/or on the course newsgroup.
- Each student must submit individual solutions for these homework problems. For all future homeworks, groups of up to three students may submit (or present) a single group solution for each problem.
- Please carefully read the course policies linked from the course web site. If you have *any* questions, please ask during lecture or office hours, or post your question to the course newsgroup. In particular:
  - Submit five separately stapled solutions, one for each numbered problem, with your name and NetID clearly printed on each page. Please do *not* staple everything together.
  - You may use any source at your disposal—paper, electronic, or human—but you ***must*** write your solutions in your own words, and you ***must*** cite every source that you use.
  - Unless explicitly stated otherwise, ***every*** homework problem requires a proof.
  - Answering "I don't know" to any homework or exam problem (except for extra credit problems) is worth 25% partial credit.
  - Algorithms or proofs containing phrases like "and so on" or "repeat this process for all n", instead of an explicit loop, recursion, or induction, will receive 0 points.

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1. (a) ***Write the sentence "I understand the course policies."***

   (b) **[5 pts]** Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. Assume reasonable but nontrivial base cases if none are given. ***Do not submit proofs***—just a list of five functions—but you should do them anyway, just for practice.

   - $A(n) = 3A(n-1) + 1$
   - $B(n) = B(n-5) + 2n - 3$
   - $C(n) = 4C(n/2) + \sqrt{n}$
   - $D(n) = 3D(n/3) + n^2$
   - $E(n) = E(n-1)^2 - E(n-2)^2$, where $E(0) = 0$ and $E(1) = 1$ *[Hint: This is easy!]*

   (c) **[5 pts]** Sort the following functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. ***Do not submit proofs***—just a sorted list of 16 functions—but you should do them anyway, just for practice.

   Write $f(n) \ll g(n)$ to indicate that $f(n) = o(g(n))$, and write $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. We use the notation $\lg n = \log_2 n$.

n	$\lg n$	$\sqrt{n}$	5^n
$\sqrt{\lg n}$	$\lg \sqrt{n}$	$5^{\sqrt{n}}$	$\sqrt{5^n}$
$5^{\lg n}$	$\lg(5^n)$	$5^{\lg \sqrt{n}}$	$5^{\sqrt{\lg n}}$
$\sqrt{5^{\lg n}}$	$\lg(5^{\sqrt{n}})$	$\lg \sqrt{5^n}$	$\sqrt{\lg(5^n)}$

2. **[CS 225 Spring 2009]** Suppose we build up a binary search tree by inserting elements one at a time from the set $\{1, 2, 3, \ldots, n\}$, starting with the empty tree. The structure of the resulting binary search tree depends on the order that these elements are inserted; every insertion order leads to a different n-node binary search tree.

   Recall that the *depth* of a leaf ℓ in a binary search tree is the number of *edges* between ℓ and the root, and the depth of a binary tree is the maximum depth of its leaves.

   (a) What is the maximum possible depth of an n-node binary search tree? Give an *exact* answer, and prove that it is correct.

   (b) *Exactly* how many different insertion orders result in an n-node binary search tree with maximum possible depth? Prove your answer is correct. *[Hint: Set up and solve a recurrence. Don't forget to prove that recurrence counts what you want it to count.]*

3. **[CS 173 Spring 2009]** A *binomial tree of order k* is defined recursively as follows:

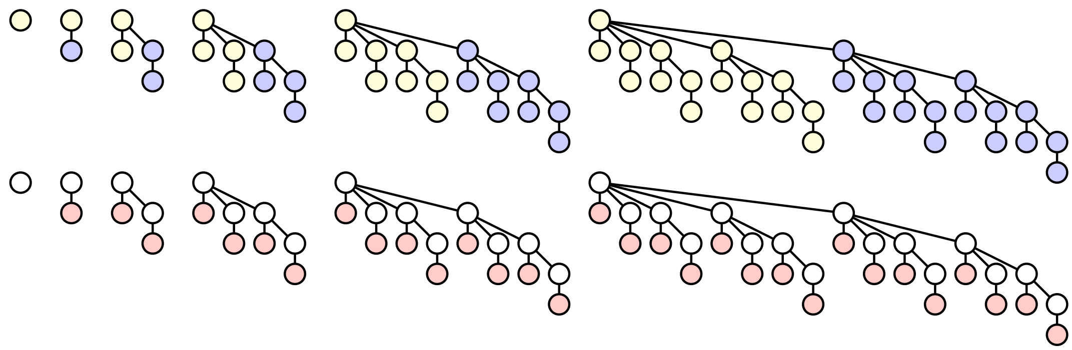
   - A binomial tree of order 0 is a single node.
   - For all $k > 0$, a binomial tree of order k consists of two binomial trees of order $k-1$, with the root of one tree connected as a new child of the root of the other. (See the figure below.)

   Prove the following claims:

   (a) For all non-negative integers k, a binomial tree of order k has exactly 2^k nodes.

   (b) For all positive integers k, attaching a leaf to every node in a binomial tree of order $k-1$ results in a binomial tree of order k.

   (c) For all non-negative integers k and d, a binomial tree of order k has exactly $\binom{k}{d}$ nodes with depth d.



Binomial trees of order 0 through 5.
Top row: the recursive definition. Bottom row: the property claimed in part (b).

4. **[CS 373 Fall 2009]** For any language $L \in \Sigma^*$, let

$$Rotate(L) := \left\{ w \in \Sigma^* \;\middle|\; w = xy \text{ and } yx \in L \text{ for some strings } x, y \in \Sigma^* \right\}$$

   For example, $Rotate(\{\texttt{OOK!}, \texttt{OOKOOK}\}) = \{\texttt{OOK!}, \texttt{OK!O}, \texttt{K!OO}, \texttt{!OOK}, \texttt{OOKOOK}, \texttt{OKOOKO}, \texttt{KOOKOO}\}$.

   Prove that if L is a regular language, then $Rotate(L)$ is also a regular language. *[Hint: Remember the power of nondeterminism.]*

5. Herr Professor Doktor Georg von den Dschungel has a 24-node binary tree, in which every node is labeled with a unique letter of the German alphabet, which is just like the English alphabet with four extra letters: **Ä**, **Ö**, **Ü**, and **ß**. (Don't confuse these with **A**, **O**, **U**, and **B**!) Preorder and postorder traversals of the tree visit the nodes in the following order:

   - Preorder: **B K Ü E H L Z I Ö R C ß T S O A Ä D F M N U G**
   - Postorder: **H I Ö Z R L E C Ü S O T A ß K D M U G N F Ä B**

   (a) List the nodes in George's tree in the order visited by an inorder traversal.

   (b) Draw George's tree.

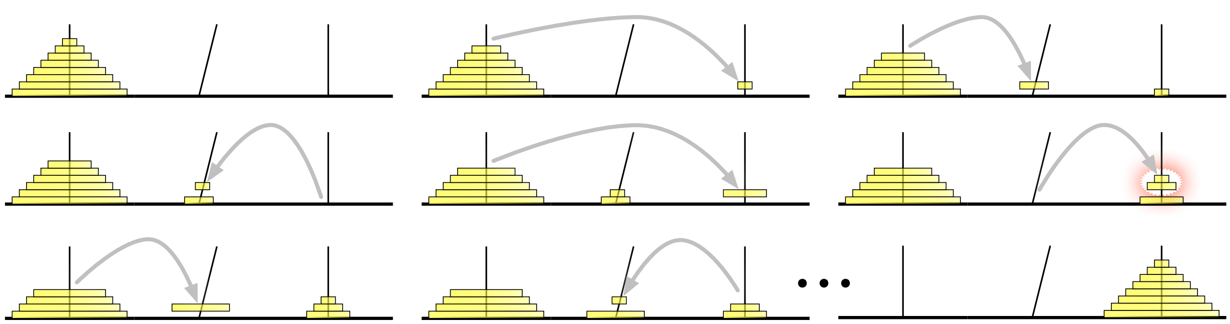
*6. ***[Extra credit]*** You may be familiar with the story behind the famous Tower of Hanoï puzzle, as related by Henri de Parville in 1884:

> In the great temple at Benares beneath the dome which marks the centre of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disc resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

A less familiar chapter in the temple's history is its brief relocation to Pisa in the early 13th century. The relocation was organized by the wealthy merchant-mathematician Leonardo Fibonacci, at the request of the Holy Roman Emperor Frederick II, who had heard reports of the temple from soldiers returning from the Crusades. The Towers of Pisa and their attendant monks became famous, helping to establish Pisa as a dominant trading center on the Italian peninsula.

Unfortunately, almost as soon as the temple was moved, one of the diamond needles began to lean to one side. To avoid the possibility of the leaning tower falling over from too much use, Fibonacci convinced the priests to adopt a more relaxed rule: ***Any number of disks on the leaning needle can be moved together to another needle in a single move.*** It was still forbidden to place a larger disk on top of a smaller disk, and disks had to be moved one at a time *onto* the leaning needle or between the two vertical needles.

Thanks to Fibonacci's new rule, the priests could bring about the end of the universe somewhat faster from Pisa then they could than could from Benares. Fortunately, the temple was moved from Pisa back to Benares after the newly crowned Pope Gregory IX excommunicated Frederick II, making the local priests less sympathetic to hosting foreign heretics with strange mathematical habits. Soon afterward, a bell tower was erected on the spot where the temple once stood; it too began to lean almost immediately.



The Towers of Pisa. In the fifth move, two disks are taken off the leaning needle.

Describe an algorithm to transfer a stack of n disks from one *vertical* needle to the other *vertical* needle, using the smallest possible number of moves. *Exactly* how many moves does your algorithm perform?