1. Suppose you are given an arbitrary directed graph \( G = (V, E) \) with arbitrary edge weights \( \ell : E \to \mathbb{R} \). Each edge in \( G \) is colored either red, white, or blue to indicate how you are permitted to modify its weight:

- You may increase, but not decrease, the length of any red edge.
- You may decrease, but not increase, the length of any blue edge.
- You may not change the length of any black edge.

The **cycle nullification** problem asks whether it is possible to modify the edge weights—subject to these color constraints—so that every cycle in \( G \) has length 0. Assume that \( G \) is strongly connected.

(a) Describe a linear program that is feasible if and only if it is possible to make every cycle in \( G \) have length 0.

**Solution:** As suggested by the hint, our linear program has a variable \( \text{dist}(v) \) for every vertex \( v \) of the graph, which represents the length of every walk from \( s \) to \( v \) with the new edge lengths.

\[
\begin{align*}
\text{maximize} & \quad 0 \\
\text{subject to} & \quad \text{dist}(v) - \text{dist}(u) \geq \ell(u \to v) \quad \text{for every red edge } u \to v \\
& \quad \text{dist}(v) - \text{dist}(u) = \ell(u \to v) \quad \text{for every white edge } u \to v \\
& \quad \text{dist}(v) - \text{dist}(u) \leq \ell(u \to v) \quad \text{for every blue edge } u \to v \\
& \quad \text{dist}(s) = 0
\end{align*}
\]

Because we only care about feasibility, the objective function doesn’t actually matter here; the objective function 0 is convenient for part (b).

(The last constraint \( \text{dist}(s) = 0 \) is actually redundant.)

**Rubric:** 5 points.

(b) Construct the dual of the linear program from part (a). [Hint: Choose a convenient objective function for your primal LP.]

**Solution:** We have a dual variable \( f(u \to v) \) for each edge \( u \to v \), corresponding to the primal constraints.

\[
\begin{align*}
\text{minimize} & \quad \sum_{u \to v} f(u \to v) \cdot \ell(u \to v) \\
\text{subject to} & \quad \sum_{u \to v} f(u \to v) - \sum_{v \to w} f(v \to w) = 0 \quad \text{for every vertex } v \neq s \\
& \quad f(u \to v) \leq 0 \quad \text{for every red edge } u \to v \\
& \quad f(u \to v) \geq 0 \quad \text{for every blue edge } u \to v
\end{align*}
\]

I called the dual variable \( f \) because the vertex constraints look like flow conservation; that’s also why I chose the primal objective vector 0.

(If we omit the redundant constraint \( \text{dist}(s) = 0 \) from the primal LP, the dual LP includes a redundant conservation constraint at \( s \).)

**Rubric:** 5 points.
(c) Give a self-contained description of the combinatorial problem encoded by the dual linear program from part (b), and prove directly that it is equivalent to the original cycle nullification problem. Do not use the words “linear”, “program”, or “dual”.

Solution: Let $H$ be the graph obtained from $G$ by inserting the reversal $v \to u$ of every white or red edge $u \to v$, defining $\ell(v \to u) = -\ell(u \to v)$ for each reversed edge, and then deleting every original red edge. The dual LP is an uncapacitated minimum-cost flow problem.

I claim that all cycles in $G$ can be nullified if and only if $H$ does not contain a negative cycle. As usual, the proof has two parts.

($\Rightarrow$) Suppose all cycles in $G$ can be nullified. Let $\ell' : E \to \mathbb{R}$ be any new length function such that all cycles in $G$ have length 0.

Fix two vertices $s$ and $v$, let $\alpha$ and $\beta$ be two walks from $s$ to $v$, and let $\gamma$ be a walk from $v$ to $s$ (which must exist because $G$ is strongly connected). The closed walks $\alpha \cdot \gamma$ and $\beta \cdot \gamma$ are composed of cycles and therefore have length zero. Thus $\alpha$ and $\beta$ have the same length, namely the negation of the length of $\gamma$. We conclude that all walks from $s$ to $v$ have the same length.

Fix an arbitrary vertex $s$ in $G$, and then for each vertex $v$, let $\text{dist}'(v)$ denote the common length of every walk from $s$ to $v$ in $G$ with respect to the new edge lengths $\ell'$. Think of each $\text{dist}'(v)$ as an estimated shortest-path distance in $H$.

To prove that $H$ has no negative cycles (with the original edge lengths $\ell$), it suffices to show that no edge in $H$ is tense. Let $u \to v$ be an arbitrary edge in $H$; there are two cases to consider:

- If $u \to v$ is a (blue or white) edge in $G$, then
  \[ \text{dist}'(v) - \text{dist}'(u) = \ell'(u \to v) \leq \ell(u \to v), \]
  which means $u \to v$ is not tense in $H$.

- If $v \to u$ is a (red or white) edge in $G$, then
  \[ \text{dist}'(u) - \text{dist}'(v) = \ell'(v \to u) \geq \ell(v \to u) = -\ell(u \to v) \]
  and thus $\text{dist}'(v) - \text{dist}'(u) \leq \ell(u \to v)$, which means $u \to v$ is not tense in $H$.

($\Leftarrow$) Now suppose $H$ does not contain a negative cycle. Then shortest-path distances in $H$ are well-defined. Add a new vertex $\hat{s}$ with zero-length edges to every vertex in $H$, and then for each vertex $v$, let $\text{dist}(v)$ denote the shortest-path distance from $\hat{s}$ to $v$ in $H$. (We need the extra vertex $\hat{s}$ because there might be no vertex that can reach every other vertex in $H$.) Finally, for every edge $u \to v$ in $G$, define $\ell'(u \to v) := \text{dist}(v) - \text{dist}(u)$.

Let $u \to v$ be an arbitrary edge in $G$. We need to verify that $\ell'(u \to v)$ is at least, at most, or equal to $\ell(u \to v)$, depending on the color of $u \to v$. There are three cases to consider.

- Suppose $u \to v$ is blue or white. Then
  \[ \ell'(u \to v) = \text{dist}(v) - \text{dist}(u) \leq \ell(u \to v) \]
  because $u \to v$ is not tense in $H$. 

– Suppose \( u \rightarrow v \) is red or white. Then

\[
\ell'(u \rightarrow v) = \text{dist}(v) - \text{dist}(u) \geq -\ell(v \rightarrow u) = \ell(u \rightarrow v)
\]

because \( v \rightarrow u \) is not tense in \( H \).

– The previous two cases imply that if \( u \rightarrow v \) is white, then \( \ell'(u \rightarrow v) = \ell(u \rightarrow v) \).

We conclude that the new lengths are consistent with the edge colors.

Finally, any cycle \( v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_{k-1} \rightarrow v_0 \) in \( G \) has length zero, because

\[
\sum_{i=0}^{k-1} \ell'(v_i \rightarrow v_{i+1 \mod k}) = \sum_{i=0}^{k-1} (\text{dist}(v_{i+1 \mod k}) - \text{dist}(v_i)) = 0.
\]

(Each term \( \text{dist}(v_i) \) appears once positively and once negatively in the second sum.)

\[\blacksquare\]

**Rubric:** 8 points = 2 for "no negative cycles in \( H \) + 3 for if proof + 3 for only if proof.

(d) Describe and analyze an algorithm to determine in \( O(EV) \) time whether it is possible to make every cycle in \( G \) have length 0, using your dual formulation from part (c).

**Solution:** We can construct the graph \( H \) in \( O(V + E) \) time, and then find negative cycles in \( H \) using a modification of the Bellman-Ford shortest-path algorithm, as described in the lecture notes, in \( O(VE) \) time.

\[\blacksquare\]

**Rubric:** 2 points.