1. Describe and analyze an efficient algorithm to find strings in labeled rooted trees. Your input consists of a *pattern string* $P[1..m]$ and a rooted *text tree* T with n nodes, each labeled with a single character. Nodes in T can have any number of children. A path in T is called a *downward path* if every node on the path is a child (in T) of the previous node in the path. Your goal is to determine whether there is a downward path in T whose sequence of labels matches the string P.

Solution (rolling hash): We use a variant of the Rabin-Karp rolling hash, as described in the lecture notes. For any string w of length m, let $h(w)$ denote its hash value. The rolling hash satisfies the following key properties:

- We can compute the hash value of any string of length m from scratch in $O(m)$ *worst-case* time.
- For any string $w[0..m]$, given the hash value $h(w[0..m-1])$ and the characters $w[0]$ and $w[m]$, we can compute the hash value $h(w[1..m])$ in $O(1)$ *worst-case* time.
- Assuming we choose the salt parameters of h randomly from a large enough range, for any two strings $w \neq w'$, both of length m, we have $\Pr[h(w) = h(w')] \leq 1/m$.

Our algorithm assumes each node v in the input tree T stores its label and a pointer to its parent, in addition to its usual child pointers. To avoid annoying boundary cases, we (pretend to) add a path of $m-1$ nodes above the root of T, each labeled with a sentinel character `@` that does not appear in the pattern string P.

For each node v in the (original) input tree T, we actally compute three values:

- $v.depth$ is the depth of v. This is used only to compute...
- $v.start$ is the label of the starting node of the unique downward path of length m ending at v. In particular, if v has depth less than m, then $v.start$ is the sentinel character `@`. This is used only to compute...
- $v.hash$ is the rolling hash value of the label of the downward path of length m ending at v.

The following algorithm computes $v.hash$ for every node v, along with the hash value of the pattern, which we call *phash*. Then for every node v such that $v.hash = phash$, the algorithm performs a brute force string comparison.

```
MatchTreePath(P[1..m], T):
    Initialize rolling hash function h
    ComputeHashes(T, m)                 ⟨⟨O(n)⟩⟩
    phash ← h(P)                        ⟨⟨O(m)⟩⟩
    for all vertices v in T
        if phash = v.hash               ⟨⟨prob < 1/m⟩⟩
            if ExactMatch(P, v)         ⟨⟨O(m)⟩⟩
                return True
    return False
```

```
ComputeHashes(T, m):
  for all nodes v in T in (depth-first) preorder
      ⟨⟨— compute depth —⟩⟩
      if v = T.root
          v.depth ← 0
      else
          v.depth ← v.parent.depth + 1
      S[v.depth] ← v.label        ⟨⟨S mirrors the recursion stack⟩⟩
      ⟨⟨— compute start symbol —⟩⟩
      if v.depth < m − 1
          v.start ← !
      else
          v.start ← S[v.depth − (m − 1)]
      ⟨⟨— compute rolling hash value —⟩⟩
      if v = T.root
          v.hash ← h(@^(m−1) · v.label)        ⟨⟨O(m)⟩⟩
      else
          compute v.hash from v.parent.hash and v.label and v.start
```

```
⟨⟨Does P match downward path ending at v?⟩⟩
ExactMatch(P, v):
  for i ← m to 1
      if v = Null and P[i] ≠ @
          return False
      if v ≠ Null and P[i] ≠ v.label
          return False
      v ← v.parent
  return True
```

This algorithm runs in $O(n + Fm)$ time, where F is the number of false matches, that is, the number of nodes v where $v.hash = phash$ but $\text{ExactMatch}(P, v) = \text{False}$. The probability of any false match is $O(1/m)$, so $E[F] = O(n/m)$. We conclude that the algorithm runs in $\boldsymbol{O(n)}$ ***expected time***. ■

Rubric: 10 points = 2 for using rolling hash + 4 for other details + 2 for analysis of false matches + 2 for time analysis. This is not the only correct $O(n)$-time algorithm. Max 5 points for $O(mn)$-time algorithm (for example, brute force).

Solution (modified KMP, 8/10): Preprocess the pattern P for KMP string matching in $O(m)$ time. Then perform a standard preorder traversal of T. Then we effectively run KMP along each root-to-leaf path in T, but using memoization to avoid scanning common prefixes of downward paths more than once. At each node v, we compute an index $v.j$ equal to the length of the longest prefix of P that is also the label of a downward path ending at v.

```
TreeKMP(P[1..m], T]):
  fail ← ComputeFailure(P)
  for all nodes v in T in preorder
      if v = T.root
          v.j ← [v.label = P[1]]
      else
          v.j ← v.parent.j + 1
          while v.j > 0 and v.label ≠ P[v.j]
              v.j ← fail[v.j]
          if v.j = m
              return True
  return False
```

Unfortunately, this algorithm does *not* run in $O(n)$ time; the amortized analysis of the number of failures doesn't generalize to trees. It's true that the number of failures *along each root-to-leaf path* is $O(n)$, but the number of failures *at each node* is unbounded. The best time bound we can guarantee is $O(mn)$, which is no better than brute force.

For example, suppose P consists of m As, and T consisting of a path of $m-1$ As whose bottom node has $n-m+1$ children, each labeled with a distinct symbol not equal to A. If we use the unoptimized KMP failure function, our algorithm fails m times at each leaf of T, so the overall running time is $\Theta(mn)$.

However, if we use the *optimized* failure function described in the lecture notes, the number of failures at each node is at most $O(\log m)$, and therefore our algorithm runs in ***$O(n \log m)$ time*** in the worst case.

The following proof of that the optimized failure function admits at most $O(\log m)$ consecutive failures is adapted from Knuth, Morris, and Pratt's paper.

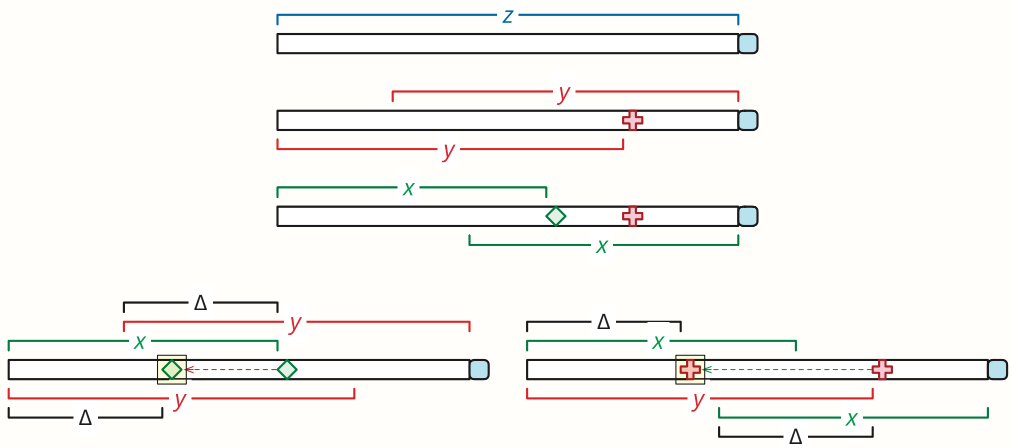
Fix an arbitrary pattern $P[1..m]$. To simplify notation, define $\ell = fail(m)$ and $k = fail(\ell)$, and consider the prefixes $z = P[1..m-1]$ and $y = P[1..\ell-1]$ and $x = P[1..k-1]$ of P. The definition of *fail* implies that both x and y are borders of z. Moreover, because the failure function is optimized, we have $P[m] \neq P[\ell]$ and $P[\ell] \neq P[k]$. (See the figure on the next page.)

For the sake of argument, suppose $\Delta = k + \ell - m \geq 0$. Then we can make two observations about the symbol $P[\Delta+1]$:

- There is an overlap of Δ symbols between the prefix y and the suffix x of z, so $x[\Delta+1] = P[\ell]$. String x is a prefix of P, so $P[\Delta+1] = x[\Delta+1] = P[\ell]$.

- There is an overlap of Δ symbols between the prefix x and the suffix y of z, so $y[\Delta+1] = P[k]$. String y is a prefix of P, so $P[\Delta+1] = y[\Delta+1] = P[k]$.

So apparently $P[\Delta+1] = P[k] = P[\ell]$. But our optimization guarantees that $P[k] \neq P[\ell]$; we have reached a contradiction! We conclude that $m \geq k + \ell$.



If $\ell \leq m/2$, then $k < \ell \leq m/2$; on the other hand, if $\ell > m/2$, then $k \leq m - \ell < m/2$. In both cases, we have $k < m/2$. Because we applied our argument to *arbitrary* patterns P, it also applies to *all prefixes of* any pattern P. We conclude that *fail*(*fail*(j)) $< j/2$ for every index j. It follows immediately that the longest failure chain in P has length at most $2\lceil \lg m \rceil$.[a] ■

[a] More careful arguments imply that the longest failure chain has length at most $\lceil \log_\phi m \rceil \approx 1.44 \lg m$, where ϕ is the golden ratio. This improved bound is actually tight for the Fibonacci strings.

Rubric: 8 points = 4 for correct algorithm + 2 for correct $O(mn)$-time analysis + 2 for correct $O(n \log m)$-time analysis with optimized fail function. **5 extra credit points** for including a proof that the optimized fail function guarantees at most $O(\log m)$ consecutive failures. (I claimed this fact in class and in the notes, so a proof isn't required for the 8 points of partial credit.)

I'm willing to believe that $O(n)$ worst-case time is possible, but I haven't figured out how.

2. Suppose we want to design an algorithm to detect the subject of a fugue. We will assume a *very* simple representation as an array $F[1\,..\,n]$ of integers, each representing a note in the fugue as the number of half-steps above or below middle C.

   (a) Describe an algorithm to find the length of the longest prefix of F that reappears later as a substring of F. The prefix and its later repetition must not overlap.

Solution: We perform a binary search for the largest value of L such that the input string contains a prefix of length L that appears later in the string.

To answer the decision problem, we search for the "pattern" string $F[1\,..\,L]$ inside the "text" string $F[L+1\,..\,n]$, using any of the linear-time algorithms described in class. Because the decision algorithm for each value of L runs in $O(n)$ time, the entire algorithm runs in $\boldsymbol{O(n \log n)}$ ***time***. ■

Rubric: 5 points: 1½ for binary search + 1½ for linear-time decision algorithm + 1 for enforcing disjointness + 1 for time analysis (if the algorithm is correct). This is not the only correct algorithm with this running time.

Solution (extra credit): Recall that a *border* of a string w is any proper prefix of w that is also a suffix of w. The Knuth-Morris-Pratt failure function is defined as follows for each index i:

$$\mathit{fail}(i) - 1 \text{ is the length of the longest border of } F[1\,..\,i-1].$$

The preprocessing phase of KMP computes $\mathit{fail}(i)$ for every index i in $O(n)$ time. To simplify notation (and avoid off-by-one errors), let $\boldsymbol{\mathit{border}(i)} = \mathit{fail}(i+1) - 1$ denote the length of the longest border $F[1\,..\,i]$. (To ensure that we compute $\mathit{border}(n) = \mathit{fail}(n+1) - 1$, add an arbitrary sentinel character $F[n+1]$ to the end of the input string.)

If we didn't require the prefix of F and its repetition to be disjoint, we would be nearly done; the longest prefix that repeats later in the string (possibly with overlap) has length $\max_{1 \le i \le n} \mathit{border}(i) = \max_{2 \le i \le n+1} \mathit{fail}(i) - 1$. But enforcing the disjointness condition requires a bit more more work.

Let's call a border ***strict*** if the equal prefix and suffix do not overlap. For example, ABAB is a strict border of ABABABAB, but ABABAB is not. For each index i, let $\boldsymbol{\mathit{border}^*(i)}$ denote the length of the longest strict border of $F[1\,..\,i]$.

Fix an index i, let $b = \mathit{border}(i)$, and let $\delta = i - b$; then $F[1\,..\,b] = F[\delta+1\,..\,i]$ is the longest border of $F[1\,..\,i]$. Assume $b > \delta$, so the prefix $F[1\,..\,b]$ and the suffix $F[\delta+1\,..\,i]$ overlap; otherwise, we immediately have $\mathit{border}^*(i) = b$. Then the shorter prefix $F[1\,..\,\delta]$ is equal to $F[\delta+1\,..\,2\delta]$, and it follows by induction that $F[1\,..\,i]$ consists of $\lfloor i/\delta \rfloor$ concatenated copies of $F[1\,..\,\delta]$, followed by a copy of the prefix $F[1\,..\,i \bmod \delta]$.

For example, consider the string

$$F[1\,..\,i] = \mathrlap{\overbrace{\phantom{\text{ABACABACABACABACABACABA}}}}\text{ABAC}\underbrace{\text{ABACABACABACABACABACABA}};$$

Here we have $i = 27$, $b = 23$, $\delta = 4$. The string consists of "$i/\delta = 5\frac{3}{4}$ copies" of the prefix ABAC, and its longest strict border has length $b - 3\delta = 11$.

Repeatedly removing the first or last δ symbols of $F[1..b]$ yields another border of $F[1..i]$, and every border of $F[1..i]$ can be obtained this way. Thus, every border of $F[1..i]$ has length $b-k\delta$ for some integer $0 \le k \le b/\delta$, and we want the longest border whose length is at most $i/2$. The smallest integer k such that $b-k\delta \le i/2$ is $k = \lceil \frac{2b-i}{2\delta} \rceil$. We conclude that

$$border^*(i) = b - \left\lceil \frac{2b-i}{2i-2b} \right\rceil \cdot (i-b),$$

where $b = border(i)$. (The ceiling expression reduces to 0 when $b \le i/2$, so the formula is correct even in that case.)

Our final algorithm has three stages: (1) Run the KMP preprocessing algorithm to compute the failure function in $O(n)$ time. (2) For each index i, compute $border^*(i)$ in $O(1)$ time. (3) Return $\max_i border^*(i)$. Altogether, our algorithm runs in ***$O(n)$ time***. ■

Solution (extra credit): We can modify the ComputeFailure algorithm from Knuth-Morris-Pratt as follows:

```
LongestSubject(F[1..n]):
  j ← 0
  j* ← 0
  longest ← 0
  for i ← 1 to n
      ⟨⟨— compute standard fail function —⟩⟩
      fail[i] ← j
      while j > 0 and F[i] ≠ F[j]
          j ← fail[j]
      j ← j + 1

      ⟨⟨— compute strict fail function —⟩⟩
      fail*[i] ← j*        ⟨⟨We don't actually need this line.⟩⟩
      while j* > 0 and (F[i] ≠ F[j*] or 2j* > i)
          j* ← fail[j*]    ⟨⟨not fail*[j]!⟩⟩
      longest ← max{longest, j*}
      j* ← j* + 1
  return longest
```

The lines in black are copied verbatim from ComputeFailure. The lines in blue compute a "strict failure" function *fail** defined as follows:

$fail^*[i] - 1$ is the length of the longest ***strict*** border of $F[1..i-1]$.

The correctness of this algorithm follows from two characterizations of strict borders:

- $F[1..j]$ is a strict border of $F[1..i]$ if and only if $F[1..j-1]$ is a strict border of $F[1..i-1]$ **and** $F[i] = F[j]$ **and** $2j \le i$.
- $F[1..j]$ is a strict border of $F[1..i]$ if and only if **either** $F[1..j]$ is the *longest* strict border of $F[1..i]$ **or** $F[1..j]$ is a border (not a *strict* border!) of the longest strict border of $F[1..i]$.

In particular, just before the last line $j^* \leftarrow j^* + 1$ is executed, $F[1..j]$ is the longest strict border of $F[1..i]$. Thus, at the end of the algorithm *longest* is the length of the longest prefix of F that appears later in F without overlapping.

The standard analysis of ComputeFailure implies that this modified algorithm runs in ***O(n) time***. Specifically: We increment j at most n times, and therefore we decrease j by setting $j \leftarrow fail[j]$ at most n times. Similarly, we increment j^* at most n times, and therefore we decrease j^* by setting $j^* \leftarrow fail[j]$ at most n times. ■

Solution (extra credit): Alternatively, we can leave ComputeFailure unchanged and modify the main loop of of Knuth-Morris-Pratt:

```
KunstDerFuge(F[1..n]):
  fail ← ComputeFailure(F)
  j ← 1
  longest ← 0
  for i ← 1 to n
      while j > 0 and (F[i] ≠ F[j] or 2j > i)
          j ← fail[j]
      longest ← max{longest, j}
      j ← j + 1
  return longest
```

This is the same as the previous solution, except that we've split the work into two for loops instead of just one. There are only three differences from KnuthMorrisPratt:

- the lines in blue, which keep track of the maximum value of j;
- the clause "or $2j > i$" in red, which ensures that $F[1..j]$ is always a *strict* border of $F[1..i]$; and
- the missing return after the while loop, because we always have $j < i \le n$.

■

Rubric: Max 7½ points (yes, out of 5) = 4 for correctly using or modifying KMP failure function + 2½ for enforcing disjointness + 1 for time analysis (if the algorithm is correct). These are not the only linear-time algorithms for this problem.

(b) Describe an algorithm to find the length of the longest prefix of F that reappears later, *possibly transposed*, as a substring of F. Again, the prefix and its later repetition must not overlap.

Solution: First we build a new difference array $\Delta F[1\,..\,n-1]$, where $\Delta F[i] = F[i+1] - F[i]$ for each index i. Any repeated prefix of F of length k, even after transposition, corresponds to a repeated prefix of ΔF with length $k-1$ without transposition. For example:

$$\begin{aligned} F &= \underbrace{3,1,4,1,5,9,2},6,5,3,1,4,\underbrace{1,-1,2,-1,3,7,0},1,4,2 \\ \implies \Delta F &= \underbrace{-2,3,-3,4,4,-7},4,-1,-2,-2,3,-3,\underbrace{-2,3,-3,4,4,-7},1,3,-2 \end{aligned}$$

So we *almost* have a reduction to part (a), but shifting from F to ΔF changes the disjointness requirement. Now we want the longest prefix of ΔF that appears as a substring later, *with a gap of at least one number* between the prefix and its repetition. For example:

$$\begin{aligned} F &= \underbrace{1,2,1,2},\underbrace{3,4,3,4},5,6 \\ \implies \Delta F &= \underbrace{+1,-1,+1},+1,\underbrace{+1,-1,+1},+1,+1 \end{aligned}$$

Adapting our first solution to part (a) is straightforward. After building ΔF, we perform a binary search for the largest value of L such that $\Delta F[1\,..\,L-1]$ is a substring of $\Delta F[L+1\,..\,n-1]$, again using any linear-time string matching algorithm for the decision problem at each step. The resulting algorithm runs in ***$O(n \log n)$ time***. ■

Rubric: 5 points = 2 for computing differences ΔF + 1 for binary search (or using part (a) as a black box) + 1 for enforcing gap between prefix and suffix + 1 for time analysis (if the algorithm is correct). This is not the only correct algorithm with this running time.

Solution: As in the previous solution, let $\Delta F[i] = F[i+1] - F[i]$ for each index i. We need to find the longest prefix of ΔF that appears as a substring later, *with a nontrivial gap* between the prefix and its repetition.

Adapting our linear-time algorithm for part (a) requires some tweaking. For each index i, and let $b = \mathit{border}(i)$ be the length of the longest border of $\Delta F[1\,..\,i]$, and let $\boldsymbol{\mathit{border}^+(i)}$ denote the length of the longest border of $\Delta F[1\,..\,i]$with a nontrivial gap between prefix and suffix. There are three cases to consider.

- If $\mathit{border}(i) < i/2$, then $\mathit{border}^+(i) = \mathit{border}(i)$.
- If $\mathit{border}(i) = i/2$, then $\mathit{border}^+(i)$ is the length of the *second* longest border of $\Delta F[1\,..\,i]$, which is $\mathit{border}(\mathit{border}(i))$.
- if $\mathit{border}(i) > i/2$, then the prefix and suffix overlap, we can follow the structural analysis in our previous solution almost verbatim; the array $\Delta F[1\,..\,i]$ is a power of the prefix $\Delta F[1\,..\,\delta]$, where $\delta = b - i$. To ensure a gap of at least one symbol, we need to change the inequality $b - k\delta \le i/2$ to $b - k\delta < i/2$. The smallest value of k that satisfies this modified constraint is $k = \left\lceil \frac{2b-i+2}{2\delta} \right\rceil$.

Summarizing, we have

$$border^+(i) = \begin{cases} border(i) & \text{if } border(i) < i/2 \\ border(border(i)) & \text{if } border(i) = i/2 \\ b - \left\lceil \dfrac{2b - i + 2}{2i - 2b} \right\rceil \cdot (i - b) & \text{otherwise} \end{cases}$$

for each index i, where $b = border(i)$ in the third case.

Our final algorithm has four stages: (1) compute the difference array ΔF; (2) run the KMP preprocessing algorithm on ΔF; (3) for each index i, compute $border^+(i)$ in $O(1)$ time; (4) return $\max_i border^+(i)$. Again, the entire algorithm runs in $\boldsymbol{O(n)}$ ***time***. ■

Solution (extra credit): As in the previous solutions, let $\Delta F[i] = F[i+1] - F[i]$ for each index i. We need to find the longest prefix of ΔF that appears as a substring later, *with a nontrivial gap* between the prefix and its repetition.

```
KunstDerFuge(F[1..n]):
  for i ← 1 to n − 1
      ΔF[i] ← F[i+1] − F[i]
  fail ← ComputeFailure(ΔF)
  j ← 1
  longest ← 0
  for i ← 1 to n
      while j > 0 and (ΔF[i] ≠ ΔF[j] or 2j ≥ i)
          j ← fail[j]
      longest ← max{longest, j}
      j ← j + 1
  return longest
```

The standard analysis of KMP implies that this algorithm runs in $O(n)$ time ■

Rubric: 7½ points (yes, out of 5) = 2 for computing ΔF + 2½ for correctly using the KMP failure function (or invoking part (a) as a black box) + 2 for enforcing gap + 1 for time analysis (if algorithm is correct). These are not the only linear-time algorithms for this problem.