Linear Programming

Inputs: a_{ij}, b_i, c_j

Outputs: $x_1, x_2, \ldots, x_d$

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{d} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{d} a_{ij} x_j \le b_i && \text{for each } i = 1 \,..\, p \\
& \sum_{j=1}^{d} a_{ij} x_j = b_i && \text{for each } i = p+1 \,..\, p+q \\
& \sum_{j=1}^{d} a_{ij} x_j \ge b_i && \text{for each } i = p+q+1 \,..\, n
\end{aligned}$$

d = # variables
dimension

n = # constraints
= number

Maximum (s,t)-Flow

$$\begin{aligned}
\text{maximize} \quad & \sum_{w} f(s\to w) - \sum_{u} f(u \to s) \\
\text{subject to} \quad & \sum_{w} f(v\to w) - \sum_{u} f(u\to v) = 0 && \text{for every vertex } v \ne s,t \\
& f(u\to v) \le c(u\to v) && \text{for every edge } u\to v \\
& f(u\to v) \ge 0 && \text{for every edge } u\to v
\end{aligned}$$

Variables: $f(u\to v)$ for each edge $u\to v$

canonical form

~~standard~~ inequality form

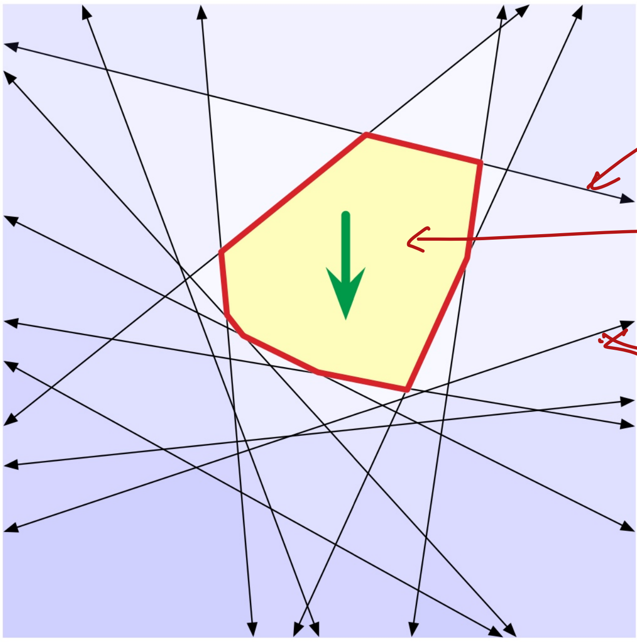
$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{d} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{d} a_{ij} x_j \le b_i && \text{for each } i = 1 \,..\, n \\
& x_j \ge 0 && \text{for each } j = 1 \,..\, d
\end{aligned}$$

constraint matrix

objective vector

offset vector

$$\boxed{\begin{aligned} \max \quad & c \cdot x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}}$$



constraint: $a_1 x_1 + a_2 x_2 \le b$

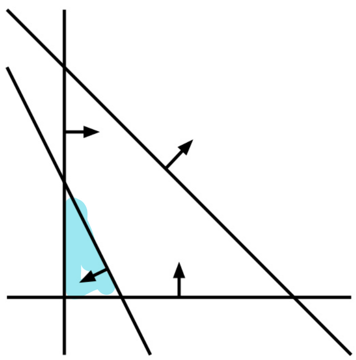
convex

feasible region / polyhedron

$\geq$

lowest point

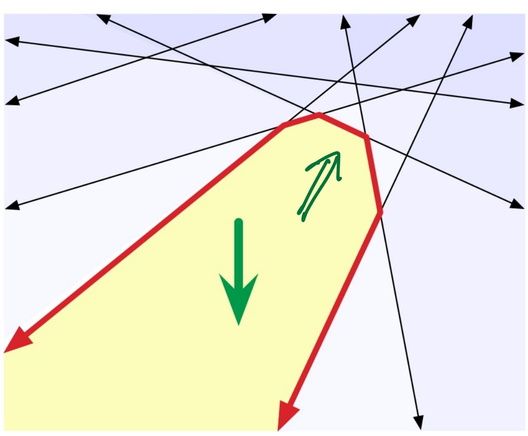
maximize	$x - y$
subject to	$2x + y \le 1$
	$x + y \ge 2$
	$x, y \ge 0$



infeasible

depends on b

does not depend on c



unbounded

depends on c

does not depend on b

variables = $dist(v)$ for all $v \in V$

givens = $\ell(u\to v)$ for all edges $u\to v$

$$\text{maximize} \quad \sum_v dist(v)$$

$$\text{subject to} \quad dist(s) = 0$$

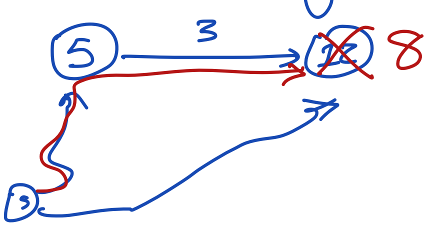
$$dist(v) - dist(u) \le \ell(u\to v) \quad \text{for every edge } u\to v$$ ← $u\to v$ is not tense

Ford's shortest path meta-algorithm

Maintain $dist(v)$ at every vertex v

Init $dist(s) = 0$ $dist(v) = \infty$

Edge $u\to v$ is tense if $dist(u) + \ell(u\to v) < dist(v)$



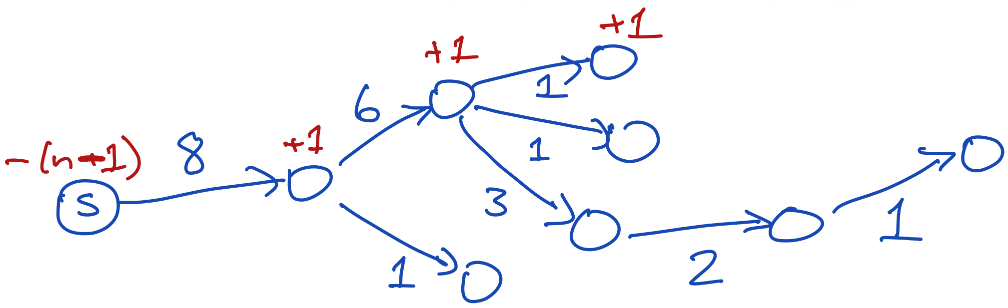
relax it → $dist(v) \leftarrow dist(u) + \ell(u\to v)$

variables

$$\text{minimize} \quad \sum_{u\to v} \ell(u\to v)\cdot \boxed{x(u\to v)}$$

$$\text{subject to} \quad \sum_u x(u\to v) - \sum_w x(v\to w) = 1 \quad \text{for every vertex } v \ne s$$

$$x(u\to v) \ge 0 \quad \text{for every edge } u\to v$$

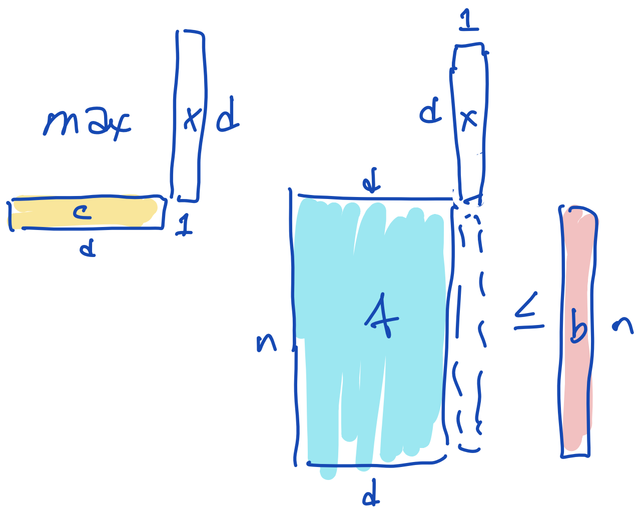


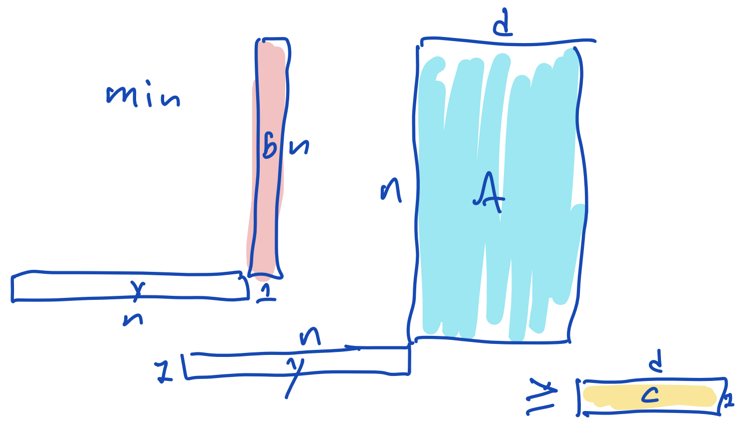
$x(u\to v)$ = # shortest paths thru $u\to v$

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_w f(s\to w) - \sum_u f(u\to s) & \\
\text{subject to} & \displaystyle\sum_w f(v\to w) - \sum_u f(u\to v) = 0 & \text{for every vertex } v \neq s, t \\
& f(u\to v) \leq c(u\to v) & \text{for every edge } u\to v \\
& f(u\to v) \geq 0 & \text{for every edge } u\to v
\end{array}
$$

$$
\begin{array}{lrl}
\text{minimize} & \displaystyle\sum_{u\to v} c(u\to v)\cdot x(u\to v) & \\
\text{subject to} & x(u\to v) + S(v) - S(u) \geq 0 & \text{for every edge } u\to v \\
& x(u\to v) \geq 0 & \text{for every edge } u\to v \\
& S(s) = 1 & \\
& S(t) = 0 &
\end{array}
$$

Primal (Π)		**Dual ($\amalg$)**		**Dual ($\amalg$)**
$\begin{aligned} \max\quad & c \cdot x \\ \text{s.t.}\quad & Ax \le b \\ & x \ge 0 \end{aligned}$	$\Longleftrightarrow$	$\begin{aligned} \min\quad & y \cdot b \\ \text{s.t.}\quad & yA \ge c \\ & y \ge 0 \end{aligned}$	$\Longleftrightarrow$	$\begin{aligned} \max\quad & -b^\top \cdot y^\top \\ \text{s.t.}\quad & -A^\top y^\top \le -c^\top \\ & y^\top \ge 0 \end{aligned}$





The Fundamental Theorem of Linear Programming. *A canonical linear program Π has an optimal solution x^* if and only if the dual linear program $\amalg$ has an optimal solution y^* such that* $\boxed{c \cdot x^* = y^* A x^* = y^* \cdot b.}$

Primal	Dual
$\max c \cdot x$	$\min y \cdot b$
$\sum_j a_{ij} x_j \le b_i$	$y_i \ge 0$
$\sum_j a_{ij} x_j \ge b_i$	$y_i \le 0$
$\sum_j a_{ij} x_j = b_i$	—
$x_j \ge 0$	$\sum_i y_i a_{ij} \ge c_j$
$x_j \le 0$	$\sum_i y_i a_{ij} \le c_j$
—	$\sum_i y_i a_{ij} = c_j$
$x_j = 0$	—

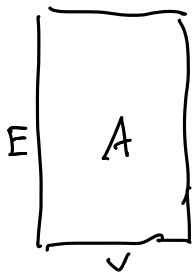
Primal	Dual
$\min c \cdot x$	$\max y \cdot b$
$\sum_j a_{ij} x_j \le b_i$	$y_i \le 0$
$\sum_j a_{ij} x_j \ge b_i$	$y_i \ge 0$
$\sum_j a_{ij} x_j = b_i$	—
$x_j \le 0$	$\sum_i y_i a_{ij} \ge c_j$
$x_j \ge 0$	$\sum_i y_i a_{ij} \le c_j$
—	$\sum_i y_i a_{ij} = c_j$
$x_j = 0$	—

Primal:

$$\begin{array}{lll} \text{maximize} & dist(t) & \\ \text{subject to} & dist(s) = 0 & \\ & dist(v) - dist(u) \le \ell(u \to v) & \text{for every edge } u \to v \end{array}$$

Variables: $dist(v)$ for every vertex $v \in V$

Constraints for every edge $u \to v$



$$A[u\to v, w] = \begin{cases} +1 & \text{if } w = v \\ -1 & \text{if } w = u \\ 0 & \text{otherwise} \end{cases}$$

Dual:

Variables: $x(u\to v)$ for each edge $u \to v$

Constraints for every vertex

$$\min \sum_{u\to v} \ell(u\to v) \cdot x(u\to v)$$

$$\text{s.t. } \sum_u x(u\to v) - \sum_w x(v\to w) = 0 \quad \text{for every } v \in V \quad v \neq s, \ v \neq t$$

$$\sum_u x(u\to t) - \sum_w (t\to w) = 1 \quad (v = t)$$

$$\begin{array}{lll} \text{minimize} & \sum_{u\to v} \ell(u\to v) \cdot x(u\to v) & \\ \text{subject to} & \sum_u x(u\to t) - \sum_w x(t\to w) = 1 & \\ & \sum_u x(u\to v) - \sum_w x(v\to w) = 0 & \text{for every vertex } v \neq s, t \\ & x(u\to v) \ge 0 & \text{for every edge } u\to v \end{array}$$