Bipartite Min Vertex Cover

Vertex cover = subset of vertices that touch every edge

NP-hard for non-bipartite graphs

$|S, T| = 4 + 2 + 5 + 3 + 5 = 19$

Cuts with finite capacity

$||(S, T)||$

$\Phi(C) = \sum_{v \in C} \$(v)$

$C = (LAT) U (RAT) U (LOS) U (RNS)$

$S = (L \setminus C) U (R \setminus C) U R S$

$T = U \setminus S$
Project selection

Input is DAG
value $\$(v)$ for every vertex $v$

Output: $S \subseteq V$ max $\$(S)$

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V = projects
E = dependencies
$u \rightarrow v$
$u$ can only be done after $v$.

Any $(S,T)$-cut $S,T$
with finite capacity

Valid selection $S \setminus \{s\}$

Claim: $\$(S \setminus s) = P - ||S,T||$

where $P = \sum_{\$(v) > 0} \$(v)$

For any $X \subseteq V$:

$\text{cost}(X) = \sum_{\$(v) < 0} \$(v) = \sum_{v \in X} c(v \rightarrow t)$

$\text{income}(X) = \sum_{\$(v) > 0} \$(v) = \sum_{v \in X} c(s \rightarrow v)$

Compute min cut $||S,T||$ in $O(n^2)$ time [Orlin]
\[
\text{profit}(x) = \text{income}(x) - \text{cost}(x) = \sum_{v \in \text{VEX}} \$v
\]

\[
P = \text{income}(V) = \text{income}(S) + \text{income}(T)
\]

\[
||S,T|| = \text{cost}(S) + \text{income}(T)
\]

\[
P - ||S,T|| = \text{income}(S) - \text{cost}(S)
\]

\[
b(v) - \text{balance} = \text{Flow in} - \text{Flow out}
\]

**Is there a Feasible Flow?**

**Necessary:** \( \sum b(v) = 0 \)

**Feasible flow \( F \) in \( G \)**

**Feasible flow \( F' \) in \( G' \)**

with value

\[
\sum b(v) > 0
\]

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with value

\( \sum b(v) > 0 \) — saturates all edges from \( S \)
Maximum Flow in network with nonzero balances

1. Feasible flow $F \rightarrow$ max flow $F'$ in $G'$
2. Maximize it $\rightarrow$ max flow $F''$ in $G_f$

\[ \text{return } F' + F'' \]