Randomized Algorithms

Library Function RANDOM(k)
$\{1, 2, 3, \ldots, k\}$
uniformly at random

Adversary

Quicksort $\rightarrow O(n^2)$

Pivot randomly $\rightarrow O(n \log n)$
Expected time

$O(n \log n)$ time with high prob.

$T(x)$ = running time on input x

$\max_{|x|=n} T(x) = T(n)$ = worstcase running time

$\max_{|x|=n} E[T(x)]$ = worst-case expected time

randomness in the algorithm.

Intro/review of discrete prob.

sample space Ω = finite/countable set

probability mass function: $\Pr : \Omega \rightarrow \mathbb{R}$

$$\Pr[\omega] \geq 0 \qquad \sum_{\omega \in \Omega} \Pr[\omega] = 1$$

Event = subset of Ω = condition/proposition

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega]$$ ← Prob of event

$A \vee B \qquad A \wedge B \qquad \neg A \qquad A \Rightarrow B \quad \ldots$

blue die + red die

$\Pr[\text{at most one } 5] = \Pr[\neg \text{ two } 5\text{s}] = 1 - \Pr[\text{two } 5\text{s}]$

$= 35/36$

Conditional probability: $\Pr[A \mid B] = \dfrac{\Pr[A \wedge B]}{P[B]}$

$\Pr[\text{red } 5 \mid \text{at least one } 5] = 6/11$

$\Pr[\text{at least one } 5 \mid \text{at most one } 5] = 2/7$

A and B are disjoint $\longleftrightarrow$ $\Pr[A \wedge B] = 0$

A and B are independent $\longleftrightarrow$ $\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B]$

HTTHTHTTHHHTH

$E[\#\text{heads}] = n/2$

$E[\#H\_] = \frac{n-1}{2}$

$E[\#HH] = \frac{n-1}{4}$

$E\left[\frac{\#HH}{\#H\_} \,\middle|\, \#H\_ > 0\right] < 1/2$

Random variable $X : \Omega \to V$ value set

$X : \Omega \to \mathbb{Z}$ "random integer" "int. random variable"

$\Pr[X = x]$ $\Pr[X \geq x]$ $\Pr[X = Y]$

Expectation: $E[X] = \sum_x x \cdot \Pr[X = x]$

$\{1, 2, 3, 4, 5, 6\}$

$E[d6] = 3\frac{1}{2}$

Conditional Exp. $E[X \mid A] = \sum_x x \cdot \Pr[X = x \mid A]$

John von Neumann 1945

Biased coin $\Pr[\text{Heads}] = p$ $\Pr[\text{Tails}] = 1 - p = q$

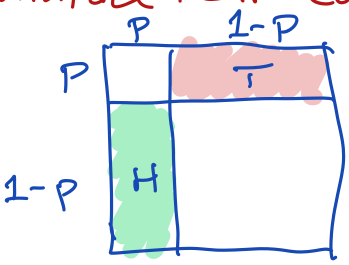
How many flips until first head? $E[\#\text{flips}] = 1/p$

$$E[\#\text{flips}] = E[\#\text{flips} \mid \text{first} = H] \cdot \Pr[\text{first} = H] + E[\#\text{flips} \mid \text{first} = T] \cdot \Pr[\text{first} = T]$$

$$= 1 \cdot p + (1 + E[\#\text{flips}])q$$

$$X = 1p + (1 + X)(1 - p) \implies X = 1/p$$

Simulate fair coin — Flip twice



HH → restart

TT → "

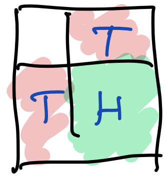
HT → "heads"

TH → "tails"

$$E[\#\text{flips}] = 2\cdot E[\#\text{trials}] = 2\cdot\frac{1}{2p(1-p)} = \boxed{\frac{1}{pq}}$$

succeeds with prob $2p(1-p)$
fails $1-2p(1-p)$

$p=1/3$



Pokémon N types of cards

You can buy random Pcard for \$1

$E[\text{cost}]$ to have at least one of each type?

Linearity of expectation

$$E[X+Y] = E[X] + E[Y]$$

$X = \#$ trials to get all N Pokémon

$X = Y_1 + Y_2 + \cdots + Y_N$

1 $Y_i = \#$trials after we have $i-1$ Pokemon to get i Pokemon

$Y_1 = 1$ $\quad E[Y_N] = N$

$$E[Y_i] = \frac{N}{N-i+1}$$

$$E[X] = \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{N} \frac{N}{N-i+1} = \sum_{j=1}^{N} \frac{N}{j} = N\sum_{j=1}^{N}\frac{1}{j}$$

$(j = N-i+1)$

$$= N H_N$$

$$\approx N \ln N$$