Randomized Algorithms

Library function $\text{RANDOM}(k)$

$\{1, 2, 3, \ldots, k\}$ uniformly at random

Adversary
- Quicksort $\Rightarrow O(n^2)$
- Pivot randomly $\Rightarrow O(n \log n)$

Expected time

$O(n \log n)$ time with high prob.

$T(x)$ = running time on input $x$

$\max_{|x|=n} T(x) = T(n) = \text{worst-case running time}$

$\max_{|x|=n} E[T(x)] = \text{worst-case expected time}$

Randomness in the algorithm.

Introduction to discrete prob.

Sample space $\Omega$ = finite/countable set

Probability mass function: $Pr: \Omega \rightarrow \mathbb{R}$

$Pr[w] \geq 0 \quad \sum_{w \in \Omega} Pr[w] = 1$

Event = subset of $\Omega = \text{condition/proposition}$

$Pr[A] = \sum_{w \in A} Pr[w] \leftarrow \text{Prob of event}$

$A \lor B \quad A \land B \quad \neg A \quad A \Rightarrow B \quad \ldots$

Blue die + red die

$Pr[\text{at most one } S] = Pr[\neg \text{two } S's] = 1 - Pr[\text{two } S's]$

$= \frac{35}{36}$

Conditional probability: $Pr[A \mid B] = \frac{Pr[A \land B]}{Pr[B]}$

$Pr[\text{red die} \mid \text{at least one } S] = \frac{6}{11}$

$Pr[\text{at least one } S] = \frac{2}{3}$

Conditional probability: $Pr[A \mid B] = \frac{Pr[A \land B]}{Pr[B]}$

$Pr[\text{at least one } S] = \frac{2}{3}$

$A$ and $B$ are disjoint $\iff Pr[A \lor B] = 0$

$A$ and $B$ are independent $\iff Pr[A \land B] = Pr[A] \cdots Pr[B]$
Random variable  \( X: \Omega \rightarrow \mathbb{V} \)  
\( X: \Omega \rightarrow \mathbb{Z} \)  "random integer"  "int. random variable"  
\( \Pr[X = x] \)  \( \Pr[X \geq x] \)  \( \Pr[X = Y] \)  
Expectation:  \( E[X] = \sum x \cdot \Pr[X = x] \)  
\( \{1, 2, 3, 4, 5, 6\} \)  \( E[6] = 3\frac{1}{2} \)  
Conditional Exp.  \( E(X|A) = \sum x \cdot \Pr[X = x|A] \)  

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John von Neumann 1945  

Biased coin  \( \Pr(\text{Heads}) = p \)  \( \Pr(\text{Tails}) = 1-p = q \)  

How many flips until first head?  \( E(\#\text{flips}) = \frac{1}{p} \)  
\( E[\#\text{flips}] = E[\#\text{flips}|\text{first} = H] \cdot \Pr(\text{first} = H) + E[\#\text{flips}|\text{first} = T] \cdot \Pr(\text{first} = T) \)  
\( = 1 \cdot p + (1+E[\#\text{flips}])q \)  
\( X = 1-p + (1+x)(1-p) \Rightarrow x = \frac{1}{p} \)  

Simulate fair coin - Flip twice  
\( \begin{array}{c|cc}
\text{H} & \text{H} & \text{T} \\
\text{P} & \text{T} & \text{H} \\
\text{1-P} & \text{T} & \text{H} \\
\end{array} \)  

\( HH \rightarrow \text{restart} \)  
\( TT \rightarrow \)  
\( HT \rightarrow \) "heads"  
\( TH \rightarrow \) "tails"
\[ E[\text{#flips}] = 2 \cdot E[\text{#trials}] = 2 \cdot \frac{1}{2p(1-p)} = \frac{1}{pq} \]

succeeds with prob \( 2p(1-p) \)
fail with prob \( 1 - 2p(1-p) \)

**Pokémon**

\( N \) types of cards

You can buy random Poké card for $1

\( E[\text{cost}] \) to have at least one of each type?

**Linearity of expectation**

\[ E[X + Y] = E[X] + E[Y] \]

\( X = \) # trials to get all \( N \) Pokémon

\( X = Y_1 + Y_2 + \ldots + Y_N \)

\( Y_i = \) # trials after we have \( i-1 \) Pokémon to get \( i \) Pokémon

\( Y_1 = 1 \quad E[Y_N] = N \)

\[ E[Y_i] = \frac{N}{N-i+1} \]

\[ E[X] = \sum_{i=1}^{N} E[Y_i] = \sum_{i=1}^{N} \frac{N}{N-i+1} = \sum_{j=1}^{N} \frac{N}{j} = N \sum_{j=1}^{N} \frac{1}{j} \]

\( \approx N \ln N \)