Reductions and NP

Lecture 2

Saturday, August 21, 2021

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How much wood would a woodchuck chuck if a woodchuck could chuck wood?

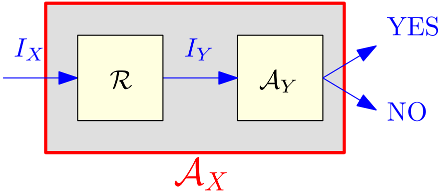
Clicker question

- (A) About as many boards as the bored Mongol hordes would hoard if the bored Mongol hordes did hoard boards in gourds.
- (B) Probably none. Woodchucks are not particularly tree-oriented. They got the name "woodchuck" from British trappers who could not quite wrap their tongues around the Cree Indian name "wuhak".
- (C) It depends on how good his dentures are.
- (D) The answer my friend is blowing in the wind.
- (E) IDK - I don't know.

Part I

Total recall...

Polynomial-time reductions



1. Algorithm is ***efficient*** if it runs in polynomial-time.
2. Interested only in polynomial-time reductions.
3. $X \leq_P Y$: Have polynomial-time reduction from problem X to problem Y.
4. $\mathcal{A}_Y$: poly-time algorithm for Y.
5. $\implies$ Polynomial-time/efficient algorithm for X.

2.1: Polynomial time reductions

Polynomial-time reductions and instance sizes

Proposition

$\mathcal{R}$: a polynomial-time reduction from X to Y.
Then, for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.
I_Y is the output of $\mathcal{R}$ on input I_X.
$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. $\square$

Unimportant remove

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

Polynomial-time Reduction

Definition

$X \leq_P Y$: ***polynomial time reduction*** from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ such that:

1. Given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y.
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$. ($|I_Y| =$ size of I_Y).
3. Answer to I_X YES $\iff$ answer to I_Y is YES.

Polynomial reductions and poly time

Polynomial reductions and poly time

Proposition

If $\boldsymbol{X} \leq_{\boldsymbol{P}} \boldsymbol{Y}$ *then a polynomial time algorithm for* $\boldsymbol{Y}$ *implies a polynomial time algorithm for* $\boldsymbol{X}$*.*

Polynomial reductions and poly time

Proposition

If $\boldsymbol{X} \leq_{\boldsymbol{P}} \boldsymbol{Y}$ then a polynomial time algorithm for $\boldsymbol{Y}$ implies a polynomial time algorithm for $\boldsymbol{X}$.

This is a ***Karp reduction***.

Composing polynomials...

A quick reminder

1. f and g monotone increasing. Assume that:
    1. $f(n) \leq a * n^b$ (i.e., $f(n) = O(n^b)$)
    2. $g(n) \leq c * n^d$ (i.e., $g(n) = O(n^d)$)

   a, b, c, d: constants.
2. $g\Big(f(n)\Big) \leq g\big(a * n^b\big) \leq c * \big(a * n^b\big)^d \leq c \cdot a^d * n^{bd}$
3. $\implies g(f(n)) = O\Big(n^{bd}\Big)$ is a polynomial.
4. **Conclusion:** Composition of two polynomials, is a polynomial.

Transitivity of Reductions

Proposition

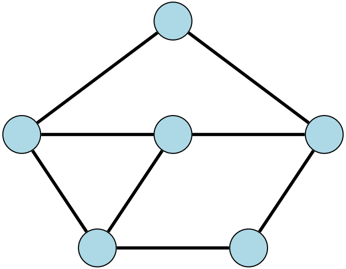
$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

1. Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.
2. To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y
3. ...show that an algorithm for Y implies an algorithm for X.

2.2: Independent Set and Vertex Cover

Vertex Cover

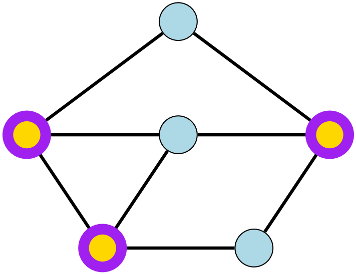
Given a graph $G = (V, E)$, a set of vertices S is:



Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

1. A ***vertex cover*** if every $e \in E$ has at least one endpoint in S.



The **Vertex Cover** Problem

Problem (**Vertex Cover**)

Input: *A graph* $\mathbf{G}$ *and integer* k.
Goal: *Is there a vertex cover of size* $\leq k$ *in* $\mathbf{G}$*?*

Can we relate **Independent Set** and **Vertex Cover**?

Relationship between...

Vertex Cover and Independent Set

Proposition

Let $G = (V, E)$ be a graph.
S is an independent set $\iff$ $V \setminus S$ is a vertex cover.

Proof:

($\Rightarrow$) Let S be an independent set

1. Consider any edge $uv \in E$.
2. Since S is an independent set, either $u \notin S$ or $v \notin S$.
3. Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
4. $V \setminus S$ is a vertex cover.

Proof continued...

$(\Leftarrow)$ Let $V \setminus S$ be some vertex cover:

1. Consider $u, v \in S$
2. uv is not an edge of $\mathbf{G}$, as otherwise $V \setminus S$ does not cover uv.
3. $\implies$ S is thus an independent set.

Independent Set $\leq_P$ Vertex Cover

1. (G, k): instance of the **Independent Set** problem.
   G: graph with n vertices. k: integer.
2. G has an independent set of size $\geq k$
   $\iff$ G has a vertex cover of size $\leq n - k$
3. (G, k) is an instance of **Independent Set** , and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
4. We conclude:
   1. **Independent Set** $\leq_P$ **Vertex Cover**.
   2. **Vertex Cover** $\leq_P$ **Independent Set**.
      (Because same reduction works in other direction.)

2.3: Vertex Cover and Set Cover

The **Set Cover** Problem

Problem (**Set Cover**)

Input: *Given a set* U *of* n *elements, a collection* $S_1, S_2, \ldots S_m$ *of subsets of* U*, and an integer* k*.*

Goal: *Is there a collection of at most* k *of these sets* S_i *whose union is equal to* U*?*

Set cover example

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

$$
\begin{array}{ll}
S_1 = \{3, 7\} & S_2 = \{3, 4, 5\} \\
S_3 = \{1\} & S_4 = \{2, 4\} \\
S_5 = \{5\} & S_6 = \{1, 2, 6, 7\}
\end{array}
$$

Solution: $\{S_2, S_6\}$ is a set cover

Set cover example

Example

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

$$\begin{array}{ll} S_1 = \{3, 7\} & S_2 = \{3, 4, 5\} \\ S_3 = \{1\} & S_4 = \{2, 4\} \\ S_5 = \{5\} & S_6 = \{1, 2, 6, 7\} \end{array}$$

Solution: $\{S_2, S_6\}$ is a set cover

Set cover example

Example

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Solution: $\{S_2, S_6\}$ is a set cover

Vertex Cover $\leq_P$ Set Cover

1. Instance of **Vertex Cover**: $\mathbf{G = (V, E)}$ and integer k.
2. Construct an instance of **Set Cover** as follows:
   1. Number k for the **Set Cover** instance is the same as the number k given for the **Vertex Cover** instance.
3. Observe that $\mathbf{G}$ has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

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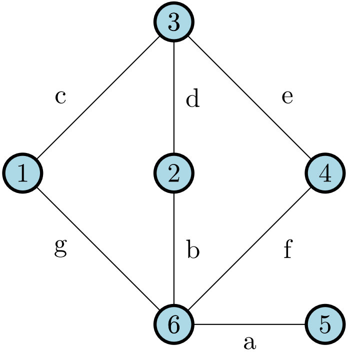
   2. $U = \mathbf{E}$.
3. Observe that $\mathbf{G}$ has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

Vertex Cover $\leq_P$ Set Cover

1. Instance of **Vertex Cover**: $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and integer k.
2. Construct an instance of **Set Cover** as follows:
   1. Number k for the **Set Cover** instance is the same as the number k given for the **Vertex Cover** instance.
   2. $U = \mathbf{E}$.
   3. We will have one set corresponding to each vertex; $S_v = \{e \mid e \text{ is incident on } v\}$.
3. Observe that $\mathbf{G}$ has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

Vertex Cover $\leq_P$ Set Cover: Example



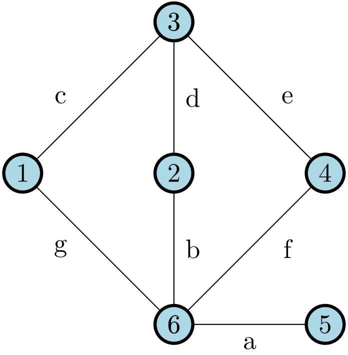
Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with

$S_1 = \{c, g\}$ $S_2 = \{b, d\}$
$S_3 = \{c, d, e\}$ $S_4 = \{e, f\}$
$S_5 = \{a\}$ $S_6 = \{a, b, f, g\}$

$\{S_3, S_6\}$ is a set cover

$\{3, 6\}$ is a vertex cover

Vertex Cover $\leq_P$ Set Cover: Example



Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with

$$S_1 = \{c, g\} \quad S_2 = \{b, d\}$$
$$S_3 = \{c, d, e\} \quad S_4 = \{e, f\}$$
$$S_5 = \{a\} \quad S_6 = \{a, b, f, g\}$$

$\{S_3, S_6\}$ is a set cover

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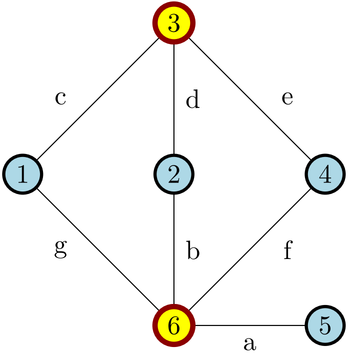
Vertex Cover $\leq_P$ Set Cover: Example



Let $U = \{a, b, c, d, e, f, g\}$, $k = 2$ with

$S_1 = \{c, g\}$ $S_2 = \{b, d\}$
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$S_5 = \{a\}$ $S_6 = \{a, b, f, g\}$

$\{S_3, S_6\}$ is a set cover

$\{3, 6\}$ is a vertex cover

Proving Reductions

To prove that $X \leq_P Y$ you need to give an algorithm $\mathcal{A}$ that:

1. Transforms an instance I_X of X into an instance I_Y of Y.
2. Satisfies the property that answer to I_X is YES $\iff$ I_Y is YES.
    1. typical easy direction to prove: answer to I_Y is YES if answer to I_X is YES
    2. typical difficult direction to prove: answer to I_X is YES if answer to I_Y is YES (equivalently answer to I_X is NO if answer to I_Y is NO).
3. Runs in ***polynomial*** time.

Summary

1. polynomial-time reductions.
    1. If $X \leq_P Y$ + have efficient algorithm for Y $\implies$ efficient algorithm for X.
    2. If $X \leq_P Y$ + no efficient algorithm for X $\implies$ **no** efficient algorithm for Y.
2. Examples of reductions between **Independent Set**, **Clique**, **Vertex Cover**, and **Set Cover**.

2.4: The Satisfiability Problem (SAT)

Propositional Formulas

Definition

Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

1. ***literal***: boolean var x_i or its negation $\neg x_i$ ($\equiv \overline{x_i}$).
2. ***clause***: disjunction literals: $x_1 \vee x_2 \vee \neg x_4$.
3. ***conjunctive normal form*** (CNF) = propositional formula which is a conjunction of clauses
   $(x_1 \vee x_2 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3}) \wedge x_5$: CNF formula.
4. A formula φ is a 3CNF:
   CNF s.t. every clause has **exactly** 3 literals.

   $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but
   $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

Satisfiability

Problem: SAT

Instance: A CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

Instance: A $3\mathrm{CNF}$ formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Satisfiability

SAT

Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

1. $(x_1 \lor x_2 \lor \neg x_4) \land (x_2 \lor \neg x_3) \land x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true
2. $(x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2) \land (x_1 \lor x_2)$ is not satisfiable.

3SAT

Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Importance of **SAT** and **3SAT**

1. **SAT**, **3SAT**: basic constraint satisfaction problems.
2. Many different problems can reduced to them: simple+powerful expressivity of constraints.
3. Arise in many hardware/software verification/correctness applications.
4. ... fundamental problem of NP-Completeness.

2.4.1: Converting a boolean formula with **3** variables to 3SAT

$z = \overline{x}$

Clicker question

Given two bits x, z which of the following **SAT** formulas is equivalent to the formula $z = \overline{x}$:

- $(\overline{z} \lor x) \land (z \lor \overline{x})$.
- $(z \lor x) \land (\overline{z} \lor \overline{x})$.
- $(\overline{z} \lor x) \land (\overline{z} \lor \overline{x}) \land (\overline{z} \lor \overline{x})$.
- $z \oplus x$.
- $(z \lor x) \land (\overline{z} \lor \overline{x}) \land (z \lor \overline{x}) \land (\overline{z} \lor x)$.

$z = x \wedge y$

Clicker question

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \wedge y$:

- $(\overline{z} \vee x \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.
- $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.
- $(\overline{z} \vee x \vee y) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y})$.
- $(z \vee x \vee y) \wedge (z \vee x \vee \overline{y}) \wedge (z \vee \overline{x} \vee y) \wedge (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y) \wedge (\overline{z} \vee \overline{x} \vee \overline{y})$.
- $(z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y)$

Converting $\mathbf{z = x \wedge y}$ to 3SAT

z	x	y					
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Converting $\mathbf{z = x \wedge y}$ to 3SAT

z	x	y	$z = x \wedge y$				
0	0	0	1				
0	0	1	1				
0	1	0	1				
0	1	1	0				
1	0	0	0				
1	0	1	0				
1	1	0	0				
1	1	1	1				

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$				
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	**0**	1	1	1
1	0	0	0	1	**0**	1	1
1	0	1	0	1	1	**0**	1
1	1	0	0	1	1	1	**0**
1	1	1	1	1	1	1	1

Converting $\mathbf{z = x \wedge y}$ to 3SAT

z	x	y	$z = x \wedge y$	$z \vee \overline{x} \vee \overline{y}$			
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	**0**	1	1	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	1	1	1	0
1	1	1	1	1	1	1	1

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$	$z \vee \overline{x} \vee \overline{y}$	$\overline{z} \vee x \vee y$		
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	0	1	1	1
1	0	0	0	1	**0**	1	1
1	0	1	0	1	1	0	1
1	1	0	0	1	1	1	0
1	1	1	1	1	1	1	1

Converting $\mathbf{z = x \wedge y}$ to 3SAT

z	x	y	$z = x \wedge y$	$z \vee \overline{x} \vee \overline{y}$	$\overline{z} \vee x \vee y$	$\overline{z} \vee x \vee \overline{y}$	
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	1	1	1	0
1	1	1	1	1	1	1	1

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$	$z \vee \overline{x} \vee \overline{y}$	$\overline{z} \vee x \vee y$	$\overline{z} \vee x \vee \overline{y}$	$\overline{z} \vee \overline{x} \vee y$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	1	1	1	**0**
1	1	1	1	1	1	1	1

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$	$z \vee \overline{x} \vee \overline{y}$	$\overline{z} \vee x \vee y$	$\overline{z} \vee x \vee \overline{y}$	$\overline{z} \vee \overline{x} \vee y$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	1	1	1	0
1	1	1	1	1	1	1	1

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$	$z \vee \overline{x} \vee \overline{y}$	$\overline{z} \vee x \vee y$	$\overline{z} \vee x \vee \overline{y}$	$\overline{z} \vee \overline{x} \vee y$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	0	1	1	1
1	0	0	0	1	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	1	1	1	0
1	1	1	1	1	1	1	1

$$\left(z = x \wedge y\right)$$

$$\equiv$$

$$(z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y)$$

Converting $\mathbf{z} = \mathbf{x} \wedge \mathbf{y}$ to 3SAT

z	x	y		
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Converting $\mathbf{z = x \wedge y}$ to 3SAT

z	x	y	$z = x \wedge y$	
0	0	0	1	
0	0	1	1	
0	1	0	1	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	0	
1	1	1	1	

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$	clauses
0	0	0	1	
0	0	1	1	
0	1	0	1	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	0	
1	1	1	1	

Converting $\mathbf{z = x \wedge y}$ to 3SAT

z	x	y	$z = x \wedge y$	clauses
0	0	0	1	
0	0	1	1	
0	1	0	1	
0	1	1	0	$z \vee \overline{x} \vee \overline{y}$
1	0	0	0	$\overline{z} \vee x \vee y$
1	0	1	0	$\overline{z} \vee x \vee y$
1	1	0	0	$\overline{z} \vee x \vee y$
1	1	1	1	

Converting $z = x \wedge y$ to 3SAT

z	x	y	$z = x \wedge y$	clauses
0	0	0	1	
0	0	1	1	
0	1	0	1	
0	1	1	0	$z \vee \overline{x} \vee \overline{y}$
1	0	0	0	$\overline{z} \vee x \vee y$
1	0	1	0	$\overline{z} \vee x \vee y$
1	1	0	0	$\overline{z} \vee x \vee y$
1	1	1	1	

$$\left(z = x \wedge y\right)$$

$$\equiv$$

$$(z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x \vee y) \wedge (\overline{z} \vee x \vee \overline{y}) \wedge (\overline{z} \vee \overline{x} \vee y)$$

Converting $\mathbf{z = x \lor y}$ to 3SAT

Simplify further if you want to

1. Using that $(x \lor y) \land (x \lor \overline{y}) = x$, we have that:
   1. $(\overline{z} \lor x \lor u) \land (\overline{z} \lor x \lor \overline{y}) = (\overline{z} \lor x)$
   2. $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) = (\overline{z} \lor y)$
2. Using the above two observation, we have that our formula $\psi \equiv \left(z \lor \overline{x} \lor \overline{y}\right) \land \left(\overline{z} \lor x \lor y\right) \land \left(\overline{z} \lor x \lor \overline{y}\right) \land \left(\overline{z} \lor \overline{x} \lor y\right)$ is equivalent to $\psi \equiv \left(z \lor \overline{x} \lor \overline{y}\right) \land \left(\overline{z} \lor x\right) \land \left(\overline{z} \lor y\right)$

Lemma

$$\left(z = x \land y\right) \quad \equiv \quad \left(z \lor \overline{x} \lor \overline{y}\right) \land \left(\overline{z} \lor x\right) \land \left(\overline{z} \lor y\right)$$

Given three bits x, y, z which of the following **SAT** formulas is equivalent to the formula $z = x \lor y$:

- $(\overline{z} \lor x \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor \overline{y})$.
- $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y})$.
- $(\overline{z} \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y})$.
- $(z \lor x \lor y) \land (\overline{z} \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y})$.
- $(z \lor x \lor y) \land (z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y}) \land (\overline{z} \lor x \lor y) \land (\overline{z} \lor x \lor \overline{y}) \land (\overline{z} \lor \overline{x} \lor y) \land (\overline{z} \lor \overline{x} \lor \overline{y})$.

Converting $z = x \lor y$ to 3SAT

z	x	y		
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Converting $\mathbf{z = x \vee y}$ to 3SAT

z	x	y	$z = x \vee y$	
0	0	0	1	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	1	
1	1	0	1	
1	1	1	1	

Converting $z = x \vee y$ to 3SAT

z	x	y	$z = x \vee y$	clauses
0	0	0	1	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	1	
1	1	0	1	
1	1	1	1	

Converting $z = x \vee y$ to 3SAT

z	x	y	$z = x \vee y$	clauses
0	0	0	1	
0	0	1	0	$z \vee x \vee \overline{y}$
0	1	0	0	$z \vee \overline{x} \vee y$
0	1	1	0	$z \vee \overline{x} \vee \overline{y}$
1	0	0	0	$\overline{z} \vee x \vee y$
1	0	1	1	
1	1	0	1	
1	1	1	1	

Converting $z = x \lor y$ to 3SAT

z	x	y	$z = x \lor y$	clauses
0	0	0	1	
0	0	1	0	$z \lor x \lor \overline{y}$
0	1	0	0	$z \lor \overline{x} \lor y$
0	1	1	0	$z \lor \overline{x} \lor \overline{y}$
1	0	0	0	$\overline{z} \lor x \lor y$
1	0	1	1	
1	1	0	1	
1	1	1	1	

$$\left(z = x \lor y\right)$$

$$\equiv$$

$$(z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y}) \land (\overline{z} \lor x \lor y)$$

Converting $\mathbf{z = x \lor y}$ to 3SAT

Simplify further if you want to

$$\left(z = x \lor y\right) \equiv (z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y}) \land (\overline{z} \lor x \lor y)$$

1. Using that $(x \lor y) \land (x \lor \overline{y}) = x$, we have that:
   1. $(z \lor x \lor \overline{y}) \land (z \lor \overline{x} \lor \overline{y}) = z \lor \overline{y}$.
   2. $(z \lor \overline{x} \lor y) \land (z \lor \overline{x} \lor \overline{y}) = z \lor \overline{x}$
2. Using the above two observation, we have the following.

Lemma

The formula $z = x \lor y$ *is equivalent to the* CNF *formula*

$$\left(z = x \lor y\right) \quad \equiv \quad (z \lor \overline{y}) \land (z \lor \overline{x}) \land (\overline{z} \lor x \lor y)$$

Converting $z = \overline{x}$ to CNF

Lemma

$$z = \overline{x} \quad \equiv \quad (z \vee x) \wedge (\overline{z} \vee \overline{x}).$$

Converting into CNF: summary

Lemma

$$z = \overline{x} \quad \equiv \quad (z \vee x) \wedge (\overline{z} \vee \overline{x}).$$

$$z = x \vee y \quad \equiv \quad (z \vee \overline{y}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x \vee y)$$

$$z = x \wedge y \quad \equiv \quad (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x) \wedge (\overline{z} \vee y)$$

Exercise...

1. **Given:**
   1. $f(x_1, \ldots, x_d)$ a boolean function
   2. Formally: $f : \{0,1\}^d \rightarrow \{0,1\}$.
2. Prove that there is CNF formula that computes f.
3. Prove that there is $3\mathrm{CNF}$ formula that computes f.

2.4.2: SAT and 3SAT

SAT $\leq_P$ 3SAT

How **SAT** is different from **3SAT**?

In **SAT** clauses might have arbitrary length: $\mathbf{1, 2, 3, \ldots}$ variables:

$$(x \vee y \vee z \vee w \vee u) \wedge (\neg x \vee \neg y \vee \neg z \vee w \vee u) \wedge (\neg x)$$

In **3SAT** every clause must have ***exactly*** **3** different literals.

Reduce from of **SAT** to **3SAT**: make all clauses to have **3** variables...

Basic idea

1. Pad short clauses so they have **3** literals.
2. Break long clauses into shorter clauses.
3. Repeat the above till we have a $\mathrm{3CNF}$.

3SAT $\leq_P$ SAT

1. **3SAT** $\leq_P$ **SAT**.
2. Because...
   A **3SAT** instance is also an instance of **SAT**.

SAT $\leq_P$ 3SAT

Claim

SAT $\leq_P$ 3SAT.

Given φ a **SAT** formula we create a **3SAT** formula φ' such that

1. φ is satisfiable iff φ' is satisfiable.
2. φ' can be constructed from φ in time polynomial in $|\varphi|$.

Idea: if a clause of φ is not of length **3**, replace it with several clauses of length exactly **3**.

SAT $\leq_P$ 3SAT

A clause with a single literal

Reduction Ideas

Challenge: Some clauses in φ # liters $\neq 3$.
$\forall$ clauses with $\neq 3$ literals: construct set logically equivalent clauses.

1. Clause with one literal: $c = \ell$ clause with a single literal.
   u, v be new variables. Consider

$$c' = (\ell \vee u \vee v) \wedge (\ell \vee u \vee \neg v) \wedge (\ell \vee \neg u \vee v) \wedge (\ell \vee \neg u \vee \neg v).$$

Observe: c' satisfiable $\iff$ c is satisfiable

SAT $\leq_P$ 3SAT

A clause with two literals

Reduction Ideas: 2 and more literals

1. Case clause with 2 literals: Let $c = \ell_1 \vee \ell_2$. Let u be a new variable. Consider

$$c' = (\ell_1 \vee \ell_2 \vee u) \wedge (\ell_1 \vee \ell_2 \vee \neg u).$$

c is satisfiable $\iff$ c' is satisfiable

Breaking a clause

Lemma

For any boolean formulas $\boldsymbol{X}$ and $\boldsymbol{Y}$ and $\boldsymbol{z}$ a new boolean variable. Then

$$\boldsymbol{X} \vee \boldsymbol{Y} \text{ is satisfiable}$$

if and only if, $\boldsymbol{z}$ can be assigned a value such that

$$(\boldsymbol{X} \vee \boldsymbol{z}) \wedge (\boldsymbol{Y} \vee \neg \boldsymbol{z}) \text{ is satisfiable}$$

(with the same assignment to the variables appearing in $\boldsymbol{X}$ and $\boldsymbol{Y}$).

SAT $\leq_P$ 3SAT (contd)

Clauses with more than 3 literals

Let $c = \ell_1 \vee \cdots \vee \ell_k$. Let $u_1, \ldots u_{k-3}$ be new variables. Consider

$$c' = (\ell_1 \vee \ell_2 \vee u_1) \wedge (\ell_3 \vee \neg u_1 \vee u_2) \\ \wedge (\ell_4 \vee \neg u_2 \vee u_3) \wedge \\ \cdots \wedge (\ell_{k-2} \vee \neg u_{k-4} \vee u_{k-3}) \wedge (\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}).$$

Claim

c *is satisfiable* $\iff$ c' *is satisfiable.*

Another way to see it — reduce size clause by one & repeat :

$$c' = (\ell_1 \vee \ell_2 \ldots \vee \ell_{k-2} \vee u_{k-3}) \wedge (\ell_{k-1} \vee \ell_k \vee \neg u_{k-3}).$$

An Example

Example

$$\varphi = (\neg x_1 \vee \neg x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ \wedge (\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1) \wedge (x_1).$$

Equivalent form:

$$\psi = (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \\ \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ \wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1) \\ \wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v) \\ \wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).$$

An Example

Example

$$\varphi = (\neg x_1 \lor \neg x_4) \land (x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_2 \lor \neg x_3 \lor x_4 \lor x_1) \land (x_1).$$

Equivalent form:

$$\begin{aligned}\psi = {} & (\neg x_1 \lor \neg x_4 \lor z) \land (\neg x_1 \lor \neg x_4 \lor \neg z) \\ & \land (x_1 \lor \neg x_2 \lor \neg x_3) \\ & \land (\neg x_2 \lor \neg x_3 \lor y_1) \land (x_4 \lor x_1 \lor \neg y_1) \\ & \land (x_1 \lor u \lor v) \land (x_1 \lor u \lor \neg v) \\ & \land (x_1 \lor \neg u \lor v) \land (x_1 \lor \neg u \lor \neg v).\end{aligned}$$

An Example

Example

$$\varphi = (\neg x_1 \vee \neg x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ \wedge (\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1) \wedge (x_1).$$

Equivalent form:

$$\begin{aligned}\psi = {} & (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \\ & \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ & \wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1) \\ & \wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v) \\ & \wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).\end{aligned}$$

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Example

$$\varphi = (\neg x_1 \vee \neg x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3 \vee x_4 \vee x_1) \wedge (x_1).$$

Equivalent form:

$$\begin{aligned}\psi = {} & (\neg x_1 \vee \neg x_4 \vee z) \wedge (\neg x_1 \vee \neg x_4 \vee \neg z) \\ & \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \\ & \wedge (\neg x_2 \vee \neg x_3 \vee y_1) \wedge (x_4 \vee x_1 \vee \neg y_1) \\ & \wedge (x_1 \vee u \vee v) \wedge (x_1 \vee u \vee \neg v) \\ & \wedge (x_1 \vee \neg u \vee v) \wedge (x_1 \vee \neg u \vee \neg v).\end{aligned}$$

Overall Reduction Algorithm

Reduction from **SAT** to **3SAT**

```
ReduceSATTo3SAT(φ):
    // φ: CNF formula.
    for each clause c of φ do
        if c does not have exactly 3 literals then
            construct c′ as before
        else
            c′ = c
    ψ is conjunction of all c′ constructed in loop
    return Solver3SAT(ψ)
```

Correctness (informal)

φ is satisfiable $\iff$ ψ satisfiable

... $\forall c \in \varphi$: new $3\mathrm{CNF}$ formula c' is equivalent to c.

Running time of converting SAT to 3SAT?

Clicker question

Let ψ be a **SAT** formula with n variables and m clauses, of total length t. Converting it to **3SAT** can be done in

- (A) $O(n+m+t)$ time.
- (B) $O\left((n+m+t)^2\right)$ time.
- (C) $O\left((n+m+t)^3\right)$ time.
- (D) $O\left((n+m+t)^4\right)$ time.
- (E) $O(1)$ time.
- (F) $O\left(t^2\right)$ time.

(Faster is better, naturally.)

What about 2SAT?

1. **2SAT** can be solved in poly time! (specifically, linear time!)
2. No poly time reduction from **SAT** (or **3SAT**) to **2SAT**.
3. If $\exists$ reduction $\implies$ **SAT**, **3SAT** solvable in polynomial time.

Why the reduction from 3SAT to 2SAT fails?

$(x \vee y \vee z)$: clause.
convert to collection of $2\mathrm{CNF}$ clauses. Introduce a fake variable α, and rewrite this as

$$(x \vee y \vee \alpha) \wedge (\neg\alpha \vee z) \qquad \text{(bad! clause with 3 vars)}$$

$$\text{or} \quad (x \vee \alpha) \wedge (\neg\alpha \vee y \vee z) \qquad \text{(bad! clause with 3 vars).}$$

(In animal farm language: **2SAT** good, **3SAT** bad.)

2.4.3: Reducing 3SAT to Independent Set

Independent Set

Problem: Independent Set

Instance: A graph G, integer k.
Question: Is there an independent set in G of size k?

3SAT $\leq_P$ Independent Set

The reduction 3SAT $\leq_P$ Independent Set

Input: Given a $3\mathrm{CNF}$ formula φ
Goal: Construct a graph G_φ and number k such that G_φ has an independent set of size k if and only if φ is satisfiable.
G_φ should be constructable in time polynomial in size of φ

1. **Importance of reduction:** Although **3SAT** is much more expressive, it can be reduced to a seemingly specialized Independent Set problem.
2. **Notice:** Handle only $3\mathrm{CNF}$ formulas (fails for other kinds of boolean formulas).

Interpreting 3SAT

There are two ways to think about **3SAT**

1. Assign 0/1 (false/true) to vars $\Longrightarrow$ formula evaluates to true. Each clause evaluates to true.
2. Pick literal from each clause & find assignment s.t. all true.

Use second view of **3SAT** for reduction.

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Use second view of **3SAT** for reduction.

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   ... Fail if two literals picked are in ***conflict***,
   e.g. you pick x_i and $\neg x_i$

Use second view of **3SAT** for reduction.

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   ... Fail if two literals picked are in ***conflict***,
   e.g. you pick x_i and $\neg x_i$

Use second view of **3SAT** for reduction.

The Reduction

1. G_φ will have one vertex for each literal in a clause
2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
3. Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
4. Take k to be the number of clauses

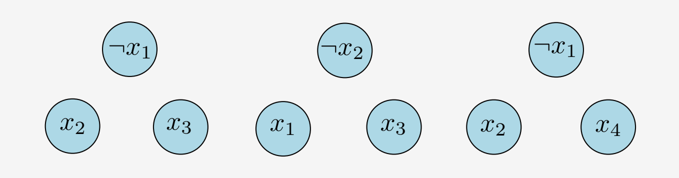

Figure: $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

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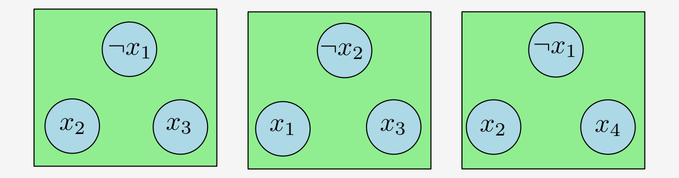

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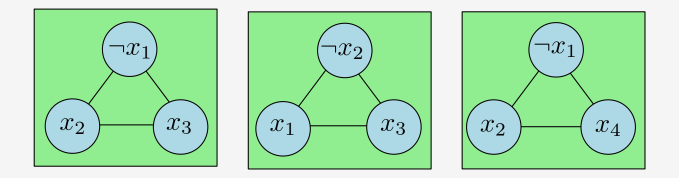

Figure: $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

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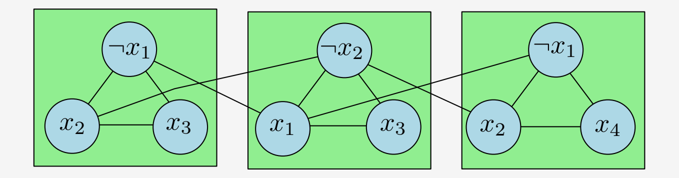

Figure: $\varphi = (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_4)$

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2. Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
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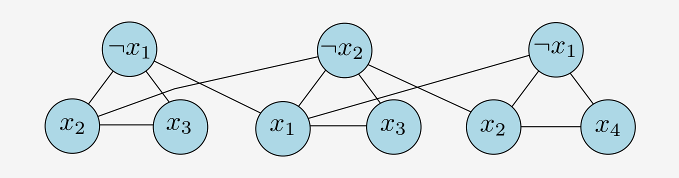

Figure: $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

Correctness

Proposition

φ is satisfiable $\iff$ G_φ has an independent set of size k
k: number of clauses in φ.

Proof.

$\Rightarrow$ a: truth assignment satisfying φ

1. Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size □

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Proposition

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Proof.

$\Rightarrow$ a: truth assignment satisfying φ

1. Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size □

Correctness (contd)

Proposition

φ is satisfiable $\iff$ G_φ has an independent set of size k (= number of clauses in φ).

Proof.

$\Leftarrow$ S: independent set in G_φ of size k

1. S must contain exactly one vertex from each clause
2. S cannot contain vertices labeled by conflicting clauses
3. Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause □

3SAT $\leq_P$ Independent Set reduction time?

Clicker question

Given an instance of **3SAT** formula with n variables, m clauses, converting it to an equivalent instance of **3SAT** takes:

- (A) $O(n + m)$.
- (B) $O(n^2 + m)$
- (C) $O(n + m^2)$
- (D) $O((n + m)^2)$
- (E) $O((n + m)^3)$