Chapter 6

Linear time algorithms

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Here, we investigate some cool algorithms for solving problems in linear time. All these algorithms use the search & prune technique – the idea is that you repeatedly reduce the input size, till it is small enough that you can

6.1. Deterministic median selection in linear time

6.1.1. Preliminaries

Definition 6.1.1 (Rank of element.). Let $X$ be an unsorted array (i.e., set) of $n$ integers. For $j$, $1 \leq j \leq n$, element of rank $j$ is the $j$th smallest element in $X$. See Figure 6.1 for an example.

The median of $X$ is the element of rank $j = [(n + 1)/2]$ in $X$.

(For simplicity, we assume all the elements of $X$ are distinct.)

Problem 6.1.2 (Selection). The input is an unsorted array $X$ of $n$ integers, and an integer $j$.

The task is to compute the $j$th smallest number in $X$ (i.e., the element of rank $j$) in $X$.

Naive algorithms for selection. The naive algorithm sorts the inputs of $X$, and returns the $j$th element in the sorted array. This takes $O(n \log n)$ time. It is natural to ask if one can avoid sorting and get a linear time algorithm.

If $j$ is small or $n - j$ is small then one can compute the $j$ smallest/largest elements in $X$ in $O(jn)$ time (by finding the minimum, deleting it, and repeating). This still fails for the median, as $j = n/2$, and the resulting running time is $O(n^2)$.
6.1.2. Divide and conquer

6.1.3. quickSelect: The divide and conquer approach

The randomized quickSelect($X, j$) works as follows:
- It “picks” a pivot element $x$ from $X$.
- Partition $X$ based on $x$ into two sets: $X_{\text{less}} = \{y \in X \mid y \leq x\}$ and $X_{\text{greater}} = \{x \in A \mid y > x\}$.
- If $|X_{\text{less}}| = j$: return $a$
- if $|X_{\text{less}}| > j$: recursively find $j$th smallest element in $X_{\text{less}}$
- if $|X_{\text{less}}| < j$: recursively find $k$th smallest element in $X_{\text{greater}}$, where $k = j - |X_{\text{less}}|$.

Running time analysis. The partitioning step takes $O(n)$ time. But how do we choose the pivot? Say the algorithm always choose the pivot to be $X[1]$, but then if $X$ is sorted in increasing order, and $j = n$. then the running time of quickSelect is $\Omega(n^2)$.

We need a better pivot. Suppose we somehow magically (using wishful thinking) chose the pivot so it is the $\ell$th smallest element in $X$, where $n/4 \leq \ell \leq 3n/4$. Namely, the pivot is approximately in the middle of $X$.

Then, it is easy to verify that
\[ n/4 \leq |X_{\text{less}}| \leq 3n/4 \quad \text{and} \quad n/4 \leq |X_{\text{greater}}| \leq 3n/4. \]

We get the following recurrence for the running time:
\[ T(n) \leq T(3n/4) + O(n), \]
and it is not hard to verify that $T(n) = O(n)$.

In the standard implementation of the algorithm, one chooses the pivot as a random element in $X$. The analysis is more complicated, and we will address it later. While this randomized algorithm can, in the worse case, run in linear time, in practice it is amazingly fast.

6.1.4. A deterministic algorithm: Median of medians

The natural approach is to try to divide the problem into many subarrays, compute the medians of the subarrays, and merge them together. The resulting algorithm is the following:

- (I) Break input $X$ into many subarrays: $L_1, \ldots, L_k$.
- (II) Find median $m_i$ in each subarray $L_i$.
- (III) Find the median $x$ of the medians $m_1, \ldots, m_k$.
- (IV) Intuition: The median $x$ should be close to being a good median of all the numbers in $X$.
- (V) Use $x$ as pivot in previous algorithm.

Example 6.1.3. The input is the array (which is already written as matrix with five rows).

```
| 75 33 13 26 83 110 | 60 120 63 30 3 41 44 | 107 30 23 91 17 | 6 110 |
| 68 24 41 26 58 57 61 | 20 52 45 13 79 86 | 91 55 66 13 103 36 60 |
| 19 40 45 111 56 74 17 95 | 96 77 29 65 36 96 93 119 9 | 61 3 9 |
| 100 3 88 47 115 107 79 39 109 20 59 25 92 81 36 10 30 113 73 116 |
| 72 58 24 16 12 69 40 24 19 92 7 65 75 41 43 117 103 38 8 20 |
```

We compute the median for each column. This results in the following numbers:
Lemma 6.1.4. We prove the following below, but for the time being assume the following lemma is correct.

See also Figure 6.2.

We compute the median of these medians. Namely, the median of the numbers in the array:

$$\begin{array}{cccccccccccccccccccc}
72 & 74 & 13 & 66 \\
61 & 60 & 65 & 30 \\
41 & 39 & 75 & 61 \\
26 & 63 & 91 & 8 \\
58 & 45 & 43 & 60 \\
\end{array}$$

The returned value is 60, and we now use it to partition After partition (pivot 60), the resulting array is:

$$\begin{array}{cccccccccccccccccccc}
41 & 24 & 26 & 56 & 17 & 40 & 24 & 42 & 30 & 7 & 60 & 77 & 81 & 63 & 61 & 107 & 115 & 111 & 72 \\
40 & 31 & 41 & 26 & 58 & 30 & 60 & 39 & 36 & 45 & 13 & 65 & 75 & 91 & 120 & 66 & 74 & 61 & 88 & 68 \\
19 & 40 & 45 & 47 & 3 & 13 & 23 & 55 & 30 & 44 & 29 & 65 & 86 & 96 & 95 & 117 & 91 & 103 & 100 & 110 \\
36 & 58 & 8 & 6 & 38 & 9 & 10 & 43 & 41 & 36 & 59 & 79 & 92 & 107 & 93 & 119 & 103 & 113 & 73 & 116 \\
\end{array}$$

The pivot has rank 57. As such, we recurse on the subarray containing the smaller elements. In this array:

$$\begin{array}{cccccccccccccccccccc}
19 & 3 & 13 & 16 & 12 & 57 & 17 & 20 & 19 & 20 & 3 & 25 \\
41 & 24 & 26 & 56 & 17 & 40 & 24 & 42 & 30 & 7 & 60 \\
40 & 31 & 41 & 26 & 58 & 30 & 60 & 39 & 36 & 45 & 13 \\
19 & 40 & 45 & 47 & 3 & 13 & 23 & 55 & 30 & 44 & 29 \\
36 & 58 & 8 & 6 & 38 & 9 & 10 & 43 & 41 & 36 & 59 \\
\end{array}$$

we are computing the element of rank 50.

Algorithm description in detail. The resulting algorithm works as follows:

(I) Partition array $X$ into $[n/5]$ lists of 5 items each.

$$L_1 = \{X[1], X[2], \ldots, X[5]\}, \quad L_2 = \{X[6], \ldots, X[10]\}, \quad \ldots, \quad L_i = \{X[5i+1], \ldots, X[5i-4]\}, \quad \ldots,$$

$$L_{[n/5]} = \{X[5\lceil n/5 \rceil - 4], \ldots, X[n]\}.$$

(II) For each $i$ find median $b_i$ of $L_i$ using brute-force in $O(1)$ time. Total $O(n)$ time

(III) Let $B = \{b_1, b_2, \ldots, b_{[n/5]}\}$

(IV) Find median $b$ of $B$

See also Figure 6.2.

6.1.5. Analysis

We prove the following below, but for the time being assume the following lemma is correct.

Lemma 6.1.4. Median of $B$ is an approximate median of $X$. That is, if $b$ is used a pivot to partition $X$, then $|X_{\text{less}}| \leq 7n/10 + 6$ and $|X_{\text{greater}}| \leq 7n/10 + 6$.

The running time recurrence is

$$T(n) \leq T([n/5]) + \max\{T(|X_{\text{less}}|), T(|X_{\text{greater}}|)\} + O(n).$$

We have that

$$T(n) \leq T([n/5]) + T([7n/10 + 6]) + O(n) \quad \text{and} \quad T(n) = O(1) \quad \text{for } n < 10.$$
Lemma 6.1.5. For $T(n) \leq T([n/5]) + T([7n/10 + 6]) + O(n)$, it holds that $T(n) = O(n)$.

Proof: We claim that $T(n) \leq cn$, for some constant $c$. We have that $T(i) \leq c$ for all $i = 1, \ldots, 1000$, by picking $c$ to be sufficiently large. This implies the base of the induction. Similarly, we can assume that the $O(n)$ in the above recurrence is smaller than $cn/100$, by picking $c$ to be sufficiently large.

So, assume the claim holds for any $i < n$, and we will prove it for $n$. By induction, we have

\[
T(n) \leq T([n/5]) + T([7n/10 + 6]) + O(n) \\
\leq c(n/5 + 1) + c(7n/10 + 6) + cn/100 \\
= cn(1/5 + 7/10 + 1/100 + 1/n + 6/n) \leq cn,
\]

for $n > 1000$. 

Claim 6.1.6. There are at least $3n/10 - 6$ elements smaller than the median of medians $b$.

Proof: There are $n/5$ medians, and as such there are $(n/5)/2 - 1$ columns where their median is strictly smaller than the median of medians. Each of these columns contribute three elements smaller than the median. There are also the two elements from the columns of the median itself. This illustrated in Figure 6.3.

Figure 6.2: Pseudo-code for deterministic median selection.

\[
\text{select}(X[1 \ldots n], j):
\begin{align*}
&\text{Form lists } L_1, L_2, \ldots, L_{\lceil n/5 \rceil} \text{ where } L_i = \{X[5i - 4], \ldots, X[5i]\} \\
&\text{Find median } b_i \text{ of each } L_i \text{ using brute-force} \\
&\text{Find median } b \text{ of } B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\} \\
&\text{Partition } X \text{ into } X_{\text{less}} \text{ and } X_{\text{greater}} \text{ using } b \text{ as pivot} \\
&\text{if } |X_{\text{less}}| = j \text{ then} \\
&\quad \text{return } b \\
&\text{if } |X_{\text{less}}| > j \text{ then} \\
&\quad \text{return select}(X_{\text{less}}, j) \\
&\text{else} \\
&\quad \text{return select}(X_{\text{greater}}, j - |X_{\text{less}}|)
\end{align*}
\]

Figure 6.3
Doing this calculation more carefully, we have that the number of elements smaller than \( b \) is
\[
3 \left\lfloor \frac{n/5 + 1}{2} \right\rfloor - 1 \geq 3n/10 - 6.
\]

This implies the following.

**Corollary 6.1.7.** We have that \( |X_{\text{greater}}| \leq 7n/10 + 6 \) and \( |X_{\text{less}}| \leq 7n/10 + 6 \).

### 6.1.6. Summary

**Theorem 6.1.8.** The algorithm \( \text{select}(X[1 \ldots n], k) \) computes in \( O(n) \) deterministic time the \( k \)th smallest element in \( X \).

On the other hand, we have (for now).

**Lemma 6.1.9.** The algorithm \( \text{quickSelect}(X[1 \ldots n], k) \) computes the \( k \)th smallest element in \( X \). The running time of \( \text{quickSelect} \) is \( \Theta(n^2) \) in the worst case.

**Question 6.1.10.** Consider the following:
- Why did we choose lists of size 5? Will lists of size 3 work?
- Write a recurrence to analyze the algorithm’s running time if we choose a list of size \( k \).

### 6.2. The lowest point above a set of lines

Let \( L \) be a set of \( n \) lines in the plane. To simplify the exposition, assume the lines are in general position:

(A) No two lines of \( L \) are parallel.
(B) No line of \( L \) is vertical or horizontal.
(C) No three lines of \( L \) meet in a point.

We are interested in the problem of computing the point with the minimum \( y \) coordinate that is above all the lines of \( L \). We consider a point on a line to be above it.

![Figure 6.4](image_url)

**Figure 6.4:** An input to the problem, the critical curve \( U_L \), and the optimal solution – the point \( \text{opt}(L) \).

For a line \( \ell \in L \), and a value \( \alpha \in \mathbb{R} \), let \( \ell(x) \) be the value of \( \ell \) at \( \alpha \). Formally, consider the intersection point of \( p = \ell \cap (x = \alpha) \) (here, \( x = \alpha \) is the vertical line passing through \( (\alpha, 0) \)). Then \( \ell(x) = y(p) \).

Let \( U_L(\alpha) = \max_{\ell \in L} \ell(\alpha) \) be the **upper envelope** of \( L \). The function \( U_L(\cdot) \) is convex, as one can easily verify. The problem asks to compute \( y^* = \min_{x \in \mathbb{R}} U_L(x) \). Let \( x^* \) be the coordinate such that \( y^* = U_L(x^*) \).
Definition 6.2.1. Let \( \text{opt}(L) = (x^*, y^*) \) denote the optimal solution – that is, lowest point on \( U_L(x) \).

Remark 6.2.2. There are some uninteresting cases of this problem. For example, if all the lines of \( L \) have negative slope, then the solution is at \( x^* = +\infty \). Similarly, if all the slopes are positive, then the solution is \( x^* = -\infty \). We can easily check these cases in linear time. In the following, we assume that at least one line of \( L \) has positive slope, and at least one line has a negative slope.

Lemma 6.2.3. Given a value \( x \), and a set \( L \) of \( n \) lines, one can in linear time do the following:

(A) Compute the value of \( U_L(x) \).

(B) Decide which one of the following happens: (I) \( x = x^* \), (II) \( x < x^* \), or (III) \( x > x^* \).

Proof: (A) Computing \( \ell(x) \), for \( x \in \mathbb{R} \), takes \( O(1) \) time. Thus computing this value for all the lines of \( L \) takes \( O(n) \) time, and the maximum can be computed in \( O(n) \) time.

(B) For case (I) to happen, there must be two lines that realizes \( U_L(x) \) – one of them has a positive slope, the other has negative slope. This clearly can be checked in linear time.

Otherwise, consider \( U_L(x) \). If there is a single line that realizes the maximum for \( x \), then its slope is the slope of \( U_L(x) \) at \( x \). If this slope is positive than \( x^* < x \). If the slope is negative then \( x < x^* \).

The slightly more challenging case is when two lines realizes the value of \( U_L(x) \). That is \( (x, U_L(x)) \) is an intersection point of two lines of \( L \) (i.e., a \text{vertex} on the upper envelope of the lines of \( L \)). Let \( \ell_1, \ell_2 \) be these two lines, and assume that \( \text{slope}(\ell_1) < \text{slope}(\ell_2) \).

If \( \text{slope}(\ell_2) < 0 \), then both lines have negative slope, and \( x^* > x \). If \( \text{slope}(\ell_1) > 0 \), then both lines have positive slope, and \( x^* < x \). If \( \text{slope}(\ell_1) < 0 \) and \( \text{slope}(\ell_1) > 0 \), then this is case (I), and we are done.

Lemma 6.2.4. Let \((x, y)\) be the intersection point of two lines \( \ell_1, \ell_2 \in L \), such that \( \text{slope}(\ell_1) < \text{slope}(\ell_2) \), and \( x < x^* \). Then \( \text{opt}(L) = \text{opt}(L - \ell_1) \), where \( L - \ell_1 = L \setminus \{\ell_1\} \)

Proof: See Figure 6.5. Since \( x < x^* \), it must be that \( U_L(\cdot) \) has a negative slope at \( x \) (and also immediately to its right). In particular, for any \( a > x \), we have that \( U_L(a) \geq \ell_2(x) > \ell_1(x) \). That is, the line \( \ell_1(x) \) is “buried” below \( \ell_2 \), and can not touch \( U_L(\cdot) \) to the right of \( x \). In particular, removing \( \ell_1 \) from \( L \) can not change \( U_L(\cdot) \) to the right of \( x \). Furthermore, since \( U_L(\cdot) \) has negative slope immediately after \( x \), it implies that minimum point can not move by the deletion of \( \ell_1 \). Thus implying the claim.

Lemma 6.2.5. Let \((x, y)\) be the intersection point of two lines \( \ell_1, \ell_2 \in L \), such that \( \text{slope}(\ell_1) < \text{slope}(\ell_2) \), and \( x^* < x \). Then \( \text{opt}(L) = \text{opt}(L - \ell_2) \).

Figure 6.5: Illustration of the proof of Lemma 6.2.4.
Proof: Symmetric argument to the one used in the proof of Lemma 6.2.4. □

Observation 6.2.6. The point \( p = \text{opt}(L) \) is a vertex formed by the intersection of two lines of \( L \). Indeed, since none of the lines of \( L \) are horizontal, if \( p \) was in the middle of a line, then we could move it and improve the value of the solution.

Lemma 6.2.7 (Prune). Given a set \( L \) of \( n \) lines, one can compute, in linear time, either:

(A) A set \( L' \subseteq L \) such that \( \text{opt}(L) = \text{opt}(L') \), and \( |L'| \leq (7/8)|L| \).

(B) A value \( x \) such that \( x^*(L) = x \).

Proof: If \( |L| = n = O(1) \) then one can compute \( \text{opt}(L) \) by brute force. Indeed, compute all the \( \binom{n}{2} \) vertices induced by \( L \), and for each one of them check if they define the optimal solution using the algorithm of Lemma 6.2.3. This takes \( O(1) \) time, as desired.

Otherwise, pair the lines of \( L \) in \( N = \lfloor n/2 \rfloor \) pairs \( \ell_i, \ell'_i \). For each pair, let \( x_i \) be the \( x \)-coordinate of the vertex \( \ell_i \cap \ell'_i \). Compute, in linear time, using median selection, the median value \( z \) of \( x_1, \ldots, x_N \). For the sake of simplicity of exposition assume that \( x_i < z \), for \( i = 1, \ldots, N/2 - 1 \), and \( x_i > z \), for \( i = N/2 + 1, \ldots, N \) (otherwise, reorder the lines and the values so that it happens).

Using the algorithm of Lemma 6.2.3 decide which of the following happens:

(I) \( z = x^* \): we found the optimal solution, and we are done.

(II) \( z < x^* \). But then \( x_i < z < x^* \), for \( i = 1, \ldots, N/2 - 1 \), By Lemma 6.2.4, either \( \ell_i \) or \( \ell'_i \) can be dropped without effecting the optimal solution, and which one can be dropped can be decided in \( O(1) \) time. In particular, let \( L' \) be the set of lines after we drop a line from each such pair. We have that \( \text{opt}(L') = \text{opt}(L) \), and \( |L'| = n - (N/2 - 1) \leq (7/8)n \).

(III) \( z > x^* \). This case is handled symmetrically, using Lemma 6.2.5. □

Theorem 6.2.8. Given a set \( L \) of \( n \) lines in the plane, one can compute the lowest point that is above all the lines of \( L \) (i.e., \( \text{opt}(L) \)) in linear time.

Proof: The algorithm repeatedly apply the pruning algorithm of Lemma 6.2.7. Clearly, by the above, this algorithm computes \( \text{opt}(L) \) as desired.

In the \( i \)th iteration of this algorithm, if the set of lines has \( n_i \) lines, then this iteration takes \( O(n_i) \) time. However, \( n_i \leq (7/8)^i n \). In particular, the overall running time of the algorithm is

\[
O\left( \sum_{i=0}^{\infty} (7/8)^i n \right) = O(n).
\]
6.3. Bottleneck edge in MST

Given an undirected graph $G = (V, E)$, with $n$ vertices and $m$ edges, and with weights $\omega(\cdot)$ on the edges. Consider the problem of computing the longest edge in the MST of $G$. One can of course compute the MST, and then compute the longest edge. However, currently no deterministic algorithm for MST is known that runs in linear time (i.e., $O(n + m)$). It turns out that there is a simple elegant algorithm that achieves linear time.

We assume that the edges of the graph $G$ has all unique weights, and the longest edge in the MST of $G$ is its bottleneck edge, denoted by $\ell_{\text{MST}}(G)$.

We assume as usually that the weights of the edges are all distinct.

6.3.1. A fast decider

**Lemma 6.3.1.** Given a graph $G$ as above with $n$ vertices and $m$ edges, and a real number $\tau$, one can decide, in $O(n \cdot m)$ time, if

(i) $\ell_{\text{MST}}(G) < \tau$,
(ii) $\ell_{\text{MST}}(G) = \tau$, or
(iii) $\ell_{\text{MST}}(G) \geq \tau$.

**Proof:** Compute the graph $G_{<\tau} = (V(G), \{uv \in E(G) \mid \omega(uv) < \tau\})$. This takes $O(n + m)$ time. If $G_{<\tau}$ has a single connected component, then $\ell_{\text{MST}}(G) > \tau$. This can be checked using BFS or DFS in $O(n + m)$ time.

Next, compute the graph $G_{\leq\tau} = (V(G), \{uv \in E(G) \mid \omega(uv) \leq \tau\})$. This takes $O(n + m)$ time. If $G_{\leq\tau}$ has a single connected component, then $\ell_{\text{MST}}(G) = \tau$. This can be checked using BFS or DFS in $O(n + m)$ time.

Otherwise, it must be that $\ell_{\text{MST}}(G) \geq \tau$.

**A naive algorithm.** One can now sort the edges of $G$ by their weights, and perform a binary search over their weights, calling the decider described above. Clearly, this would compute the bottleneck weight (and thus the edge) in $O((n + m) \log m)$ time.

6.3.2. A search and prune algorithm

The algorithm first verifies that the input graph is connected. If not, it immediately rejects it.

Otherwise, the algorithm would compute the edge realizing the median edge weight in $G$. This can be done in $O(m)$ time, and let $e$ be this edge. Calling the decider we decide how $\omega(e)$ relates to the weight of the bottleneck edge in the MST. There are three possibilities:

(I) $\ell_{\text{MST}}(G) < \tau$. None of the edges longer than $\tau$ can appear in the MST. We might as well throw them all away. Let $G''$ be the resulting graph. This graph has at most $m/2$ edges, has the same bottleneck edge is $G$, and computing it takes $O(n + m)$ time.

(II) $\ell_{\text{MST}}(G) > \tau$: The algorithm computes the graph $G_{\leq\tau} = (V(G), \{uv \in E(G) \mid \omega(uv) \leq \tau\})$. We know that each connected component of this graph would be spanned by a tree in the Kruskal algorithm for computing MST after inserting all the edges of weights smaller than $\tau$. As such, we collapse each connected component of $G_{\leq\tau}$ into a single vertex. An edge $uv \in E(G)$ connecting two different connected components of $G_{\leq\tau}$ is added to the graph, as connecting the two connected components. Note that we might have parallel edges, but it is straightforward to use bucket sort to sort the edges, so that all the edges connecting the same connected components are grouped
together. We then keep only the cheapest edge. Let $G'$ be the resulting graph. Importantly, the bottleneck edge in $G$ and $G'$ are the same, and computing $G'$ took $O(n + m)$ time. Furthermore, $G'$ has at most $m/2$ edges.

(III) $t_{\text{MST}}(G) = \tau$: We are done. Yey!

If the algorithm is not done, it continues recursively on the $G'$ or $G''$ (depending on the case). The key observation is that in either case, these two graphs are connected, and thus the number of edges dominate the number of vertices. We thus have the recurrence

$$T(m) = O(m) + T(m/2),$$

and the solution to this recurrence is $O(m)$.

**Theorem 6.3.2.** Given an undirected graph $G = (V, E)$, with $n$ vertices and $m$ edges, and with weights $\omega(\cdot)$ on the edges. The bottleneck edge of the MST can be computed in $O(n + m)$ time.

### 6.4. Bibliographical notes

The beautiful median of medians algorithm of Section 6.1 is from Blum et al. [BFP+73] (there are four Turing award winners among the authors of this paper).

The algorithm presented in Section 6.2 is a simplification of the work of Megiddo [Meg84]. Megiddo solved the much harder problem of solving linear programming in constant dimension in linear time, The algorithm presented is essentially the core of his basic algorithm.

**Bibliography**
