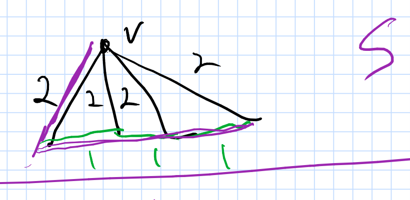
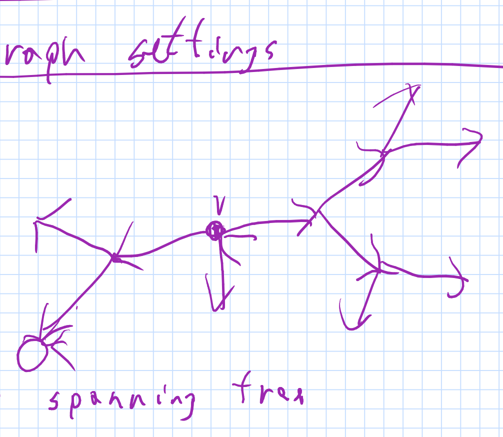
26: Directed MST

12/2/2021

G: graph
Positive weights on the edges.
r: root
Root would like to broadcast message to all nodes in the graph.



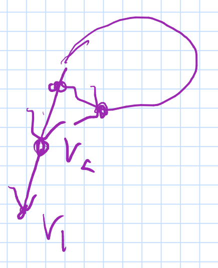
Directed graph settings



DST – Directed spanning tree

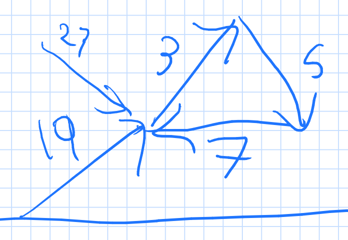
Claim In a DST for a root r:
- r has no incoming edge
- every vertex has exactly one incoming edge
- Any directed graph $H \subseteq G$ with the above properties that does not have a cycle is a DST.

proof
$v \in V \setminus \{r\}$



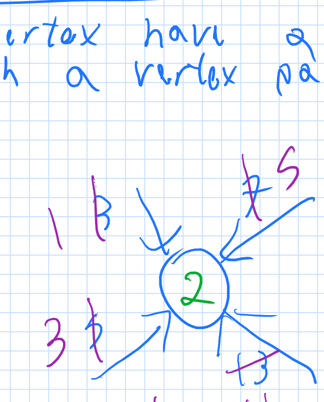
F: set of edges picked by picking for every vertex (except root) the cheapest edge incoming into it

(V, F): Frame
Two possibilities
- (V, F) contains no cycle. We are done! as (V, F) is the DMST!
- (V, F has a cycle C.



$\forall e \quad w(e) \geq 0$ weight / price e
$\forall u \to v \quad w(u \to v) \geq 0$

For every vertex have a variable y_v which is how much a vertex pays.



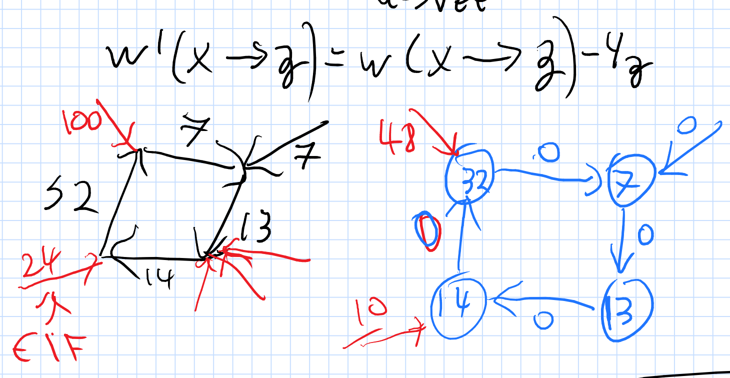
For every $u \to v$ in E set its new price to
$w'(u \to v) = w(u \to v) - y_v$

Claim
T is a DMST for W $\Leftrightarrow$ T is a DMST for w'. $\forall$ $w(u \to v) \geq 0$, $w'(u \to v) \geq 0$

proof
S be a DST.

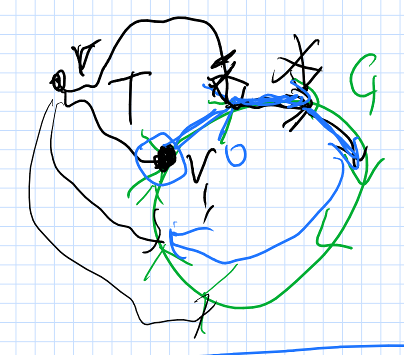
$$\underbrace{w'(S)}_{\text{MDST}} = \sum_{u \to v \in S} w'(u \to v) = \sum_{u \to v \in S} (w(u \to v) - y_v)$$
$$= \sum_{u \to v \in S} w(u \to v) - \sum_{v \in V - r} y_v$$
$$= w(S) - \sum_{v \in V - r} y_v$$

(V, F): Frame of G
- If F has no cycles we are done
- If F has a cycle C then
  $\forall v \in C \quad y_v = \min_{u \to v \in F} w(u \to v)$
  $w'(x \to z) = w(x \to z) - y_z$



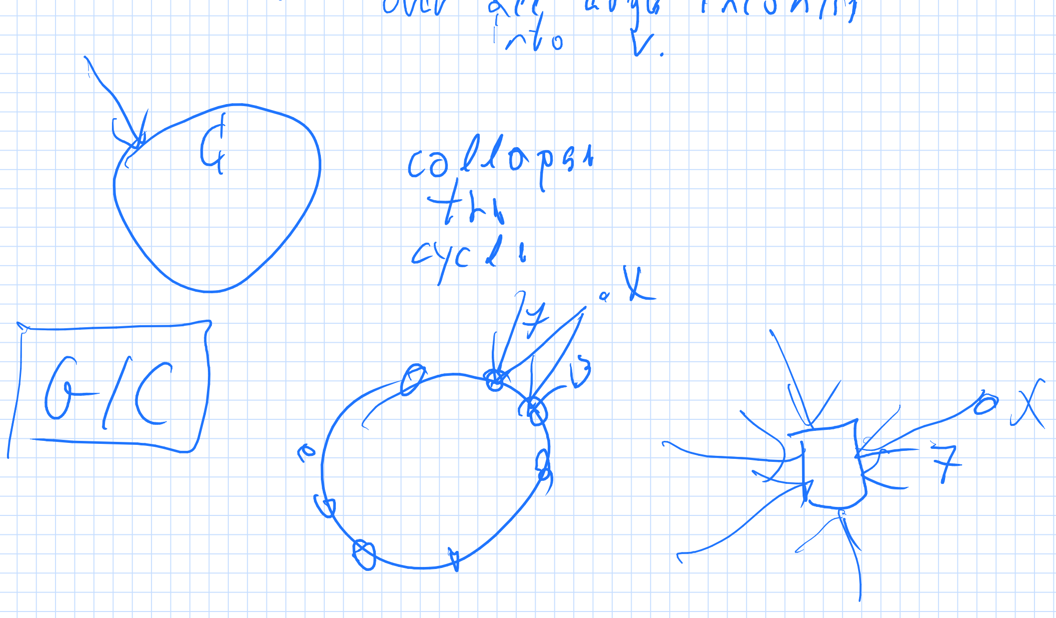
Lemma Let F be a frame, let C be a cycle s.t. $\forall u \to v \in C \quad w'(u \to v) = 0$
And let T be a DMST of G under w'
$\Rightarrow$ one can rewrite T s.t. it uses all edges of C except one edge, and the new tree $\hat{T}$ has $w'(\hat{T}) = w'(T)$.
And there is a single incoming edge into a vertex of C.

proof
T

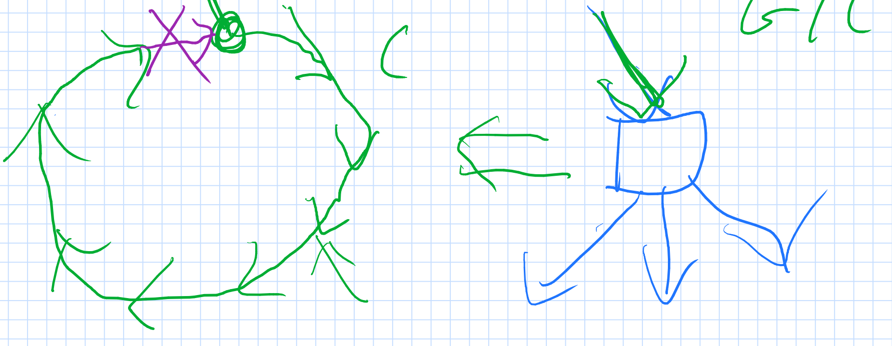


Back to algorithm

- If there is a cycle C in F
  - Adjust costs as described above
    $\forall u \to v \in F \quad v \in V(C)$
    $w'(u \to v) = w(u \to v) - y_v$
    y_v = min cost of $w(u \to v)$ over all edges incoming into v.



Compute DMST for G/C recursively.



$O(n(m+n))$ Running time.

$O(n \log n + m)$
$O((n+m) \log m)$