Vertex cover

\( G = (V, E) \) undirected graph

Compute smallest \( x \subseteq V \) s.t.

\[\forall e = (u, v) \in E \quad x_u \cup x_v \geq 1\]

s.t. \( x \) is a VC of \( G \).

\[\text{LP} \]

\[
\begin{align*}
\min & \quad \sum_{u \in V} x_u \\
\text{s.t.} & \quad \forall e = (u, v) \in E, x_u + x_v \geq 1 \\
& \quad \forall u \in V, x_u \geq 0
\end{align*}
\]

\[\text{opt}\text{-LP} = \min x \text{ VC of } G\]

\[\text{Integral solution} \geq \text{Fractional solution}\]

\[
\begin{align*}
\text{opt}\text{-VC} = \min x \text{ VC of } G \\
\text{must be} & \text{ integral or } x \geq 1/2
\end{align*}
\]

\[
\begin{align*}
C = \{ (v, w) | x_v \geq 1/2 \} \\
\text{Claim: } c \in \text{ a VC of } G.
\end{align*}
\]

Proof:

\[
\begin{align*}
\text{cost}(C) = \sum_{v \in C} x_v \\
& \leq 2 \cdot \text{opt}\text{-VC}
\end{align*}
\]

Theorem

Weighted VC can be \( 2 \) approximated by solving LP once, and doing rounding as follows.

Set Cover

\[ (U, F) \]

\[ C = \text{cover} \min \sum_{F 

\in C} x_F = 2^{\frac{1}{2}}
\]

\[
\begin{align*}
\text{Proof: } & \text{ground } v \text{ to } \min \text{ set } w \text{ in the set with probability } \frac{3}{8}. \\
& 0 \leq \text{cost}(C) = \sum_{F \in C} x_F \\
\text{claim: } & \text{prob } \text{cost}(C) \geq 1/8 \\
\text{prob: } & \text{not cover } v, X \text{ not cover } v
\end{align*}
\]

\[
\begin{align*}
\text{Claim: } \text{prob } \text{cost}(C) = 1/8 \\
& = \left( 1 - \frac{3}{8} \right) \text{prob } \text{not cover } v, X \text{ cover } v
\end{align*}
\]

\[
\begin{align*}
& = \left( 1 - \frac{3}{8} \right) \cdot \exp(-\frac{3}{8}) = \exp(-\frac{3}{8}) = \exp(-\frac{1}{8}) = \frac{1}{2}
\end{align*}
\]

Let \( R_1, R_2, \ldots \) be the same component in \( u = O(10\log n) \) rounds.

Claim: \( V \) is a VC of \( G \) with prob \( 1 - 1/n \).

Proof:
Claim
\[ \forall \mathcal{R}_i \text{ is a cover of the ground set } \mathcal{V}. \]

Proof
\[
P(C_i \text{ is not covered}) = \prod_{i=1}^{u^2} P[R_j \text{ does not cover } v_i] \leq \left( \frac{1}{2u} \right)^{u^2} = \frac{1}{n^{u^2}} \leq 1 - n^{o(1)}. \]

\[
P[\text{event}] = \bigvee_i P[C_i \text{ not covered}] \leq n \cdot \frac{1}{n^{o(1)}} = \frac{1}{n^{o(1)}}. \]

Claim
\[
E[\bigcup_{i=1}^{u^2} R_j] = \sum_{i=1}^{u^2} x_i^* = \text{opt} + U
\]
\[
\implies E[\bigcup_{i=1}^{u^2} R_j] \leq E\left[ \sum_{i=1}^{u^2} \frac{1}{2^i} |R_j| \right] = \frac{u}{2} E[|R_j|]
\]
\[
= \text{opt} + U
\]
\[
= O(\text{opt} + \text{deg} \cdot \text{log } n). \]

Conclusion

\[ \boxed{} \]