Matching

Given $G = (V, E)$, Matching $M \subseteq E$ s.t all edges in $M$ are vertex-disjoint.

$\text{Given: Graph } G = (V, E)$

$\text{Find: Matching } M \text{ of maximum size}$

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\[ M + M' \text{ is a max-size matching } M' \]

\[ M \Delta M' = (M \setminus M') \cup (M' \setminus M) \]

symmetric difference

$\Rightarrow \text{ set of "alternating paths"}$

$\Rightarrow \text{ set of "alternating cycles"}$
augmenting paths

\[ M \text{ is not a maximum matching} \Rightarrow \exists \text{ augmenting path} \]

\[ M' = M \Delta P \]

\[ \exists \text{ augmenting path in } G \]

\[ M \text{ is a maximum matching} \]

\[ \exists \text{ no augmenting path in } G \]

**Algorithm**

- maintain a matching \( M \)
- obtain a matching \( M' \)
- \( |M'| = |M| + 1 \)
- \( T(M) \)

\[ \exists \text{ max-size matching in } O(nm) \]

**Bipartite Graph**

\[ G = (A \cup B, E) \]
Given a set $M = \{m_1, m_2, \ldots, m_m\}$ of $n$ men and a set $W = \{w_1, w_2, \ldots, w_n\}$ of $n$ women, each man $m_i$ has a preference order over $W$

$$w_2 \succ_{m_i} w_3 \succ_{m_i} w_1 \succ_{m_i} w_n$$

Each woman $w_i$ has a preference order over $M$

$$m_5 \succ_{w_i} m_6 \succ_{w_i} m_7 \succ_{w_i} m_8 \succ_{w_i} \cdots \succ_{w_i} m_1$$

Thus a max-size matching in breadth-first search can be found in $O(nm)$ time.
**Blocking pair:** Given a matching \( M \), \((m_i, w_j)\) form a blocking pair iff both \( m_i \) and \( w_j \) prefer each other over their matched partners.

**Stable matching:** Matching \( M \), s.t. there are no blocking pairs.

**Goal:** Find a stable matching \( M \)

```
\[
\begin{array}{ccc}
1 & A & \quad A \succ_{1,2} B \\
2 & \quad A & \quad 1 \succ_{4,3} B \\
\end{array}
\]

MEN    WOMEN

[Diagram showing blocking pair]

<table>
<thead>
<tr>
<th>Preference</th>
<th>men</th>
<th>woman</th>
</tr>
</thead>
<tbody>
<tr>
<td>A \succ B</td>
<td>1</td>
<td>\succ 1</td>
</tr>
<tr>
<td>A \succ B</td>
<td>2</td>
<td>\succ 2</td>
</tr>
</tbody>
</table>
```

```
\[
\begin{array}{c}
1 \quad 1 \\
2 \quad B \\
\end{array}
\]

[Diagram showing male-preferred stable matching]

```
\[
\begin{array}{c}
1 \quad 2 \\
1 \quad B \\
2 \quad 2 \\
\end{array}
\]

[Diagram showing female-preferred stable matching]
```
While there are unmatched men,

each unmatched man $m_i$ proposes their best woman $w_j$, whom they have not proposed before.

If $w_j$ prefers $m_i$ over her current matched partner,

match $(m_i, w_j)$

unmatch $w_j$ with her previous partner.

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**Gale-Shapley Algorithm (GS)**

**Preferences**

<table>
<thead>
<tr>
<th>men</th>
<th>women</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A &gt; B$</td>
<td>$1 &gt; 2$</td>
</tr>
<tr>
<td>$B &gt; A$</td>
<td>$1 &gt; 2$</td>
</tr>
</tbody>
</table>

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**Stable Matching**

Round 1

1. $A \to D$
2. $B \to C$
3. $C \to A$

1. $A \to 1$
2. $B \to 2$
3. $C \to 3$
4. $D \to 4$

1. $A \to 1$
2. $B \to 2$
3. $C \to 3$
4. $D \to 4$
Correctness → outputs a stable matching

Claim: every man is matched.
Assume otherwise

↓
→ man rejected by all women

↓
→ all women are matched.

why Stable? M is the output of GS
Assume otherwise

i

O

M

O

j

↓

i' O

j' 

(i, i') is a blocking pair
\((s, 0')\) = blocking pair
\[s, i'\]

\[i \geq i' \Rightarrow (s, i')\] is not a blocking pair

**Running time:** \(O(\text{# proposals made})\)

**Thm:** At the end of step 8, every man is matched to the best woman he can be matched in any stable matching.

**Stable pairs:** \((u, j)\) form a stable pair if \(F \& u \leq j\)
matching value $E$ & $j$ are matched

Execution $E$ of the CA algorithm

$M'$ is another stale match

Reflection $(i', j) = \text{rejection of a stale pair}$

First time wrong in $E$, but a rejection of a stale pair has happened
\[ z \quad i \quad i' \quad \bar{z} \quad i'' \]

ALWAES PROPOSAL