L20: Linear programming

11/2/2021

LP

$$\boxed{\begin{array}{l} x_1, x_2, \dots, x_d \\ \min \sum c_i x_i \\ \sum a_{ij} x_i \le b_i \end{array}} \quad i = 1, \dots, m$$

$Q: \exists \ (x_1, x_2, \dots, x_d)$

$d = 2$

$$\boxed{\begin{array}{l} \max 7x + 7y \\ x + y \le 1 \\ x - y \ge 7 \end{array}}$$

$x + y = 1$

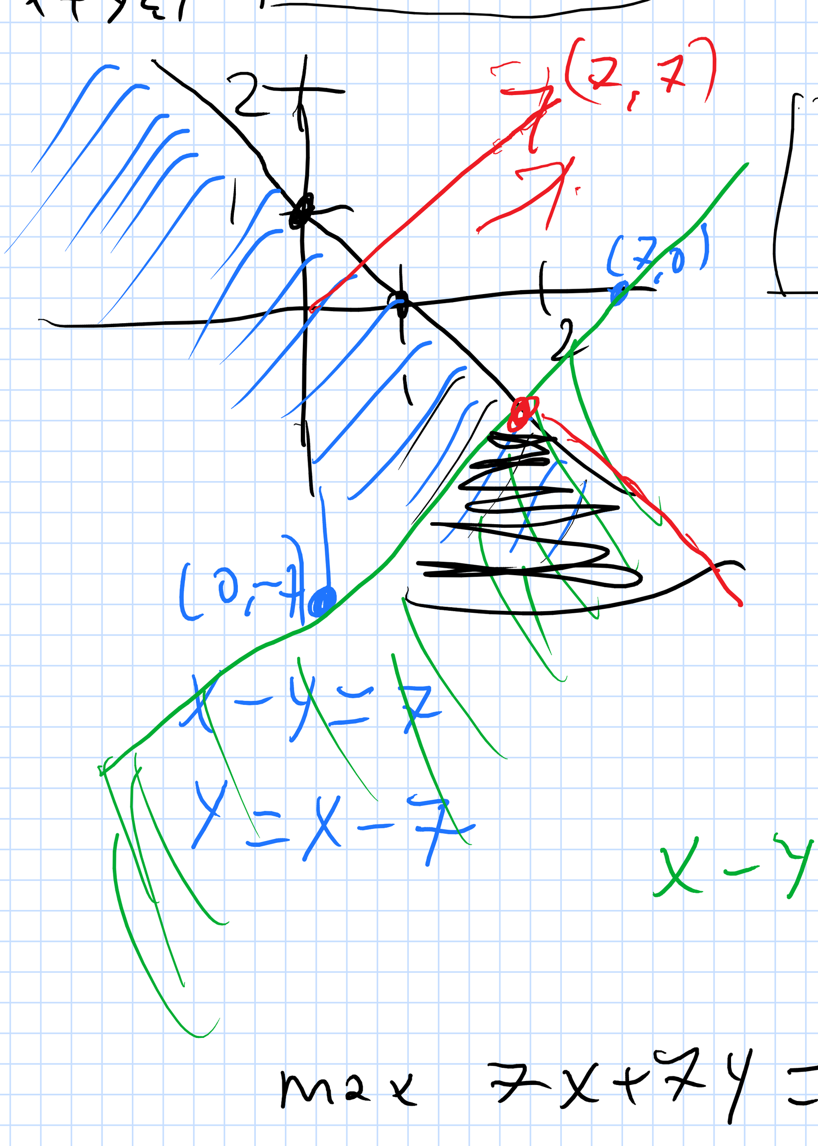
$x + y < 1$

$\max \sum a_i x_i$

$\min \sum -a_i x_i$

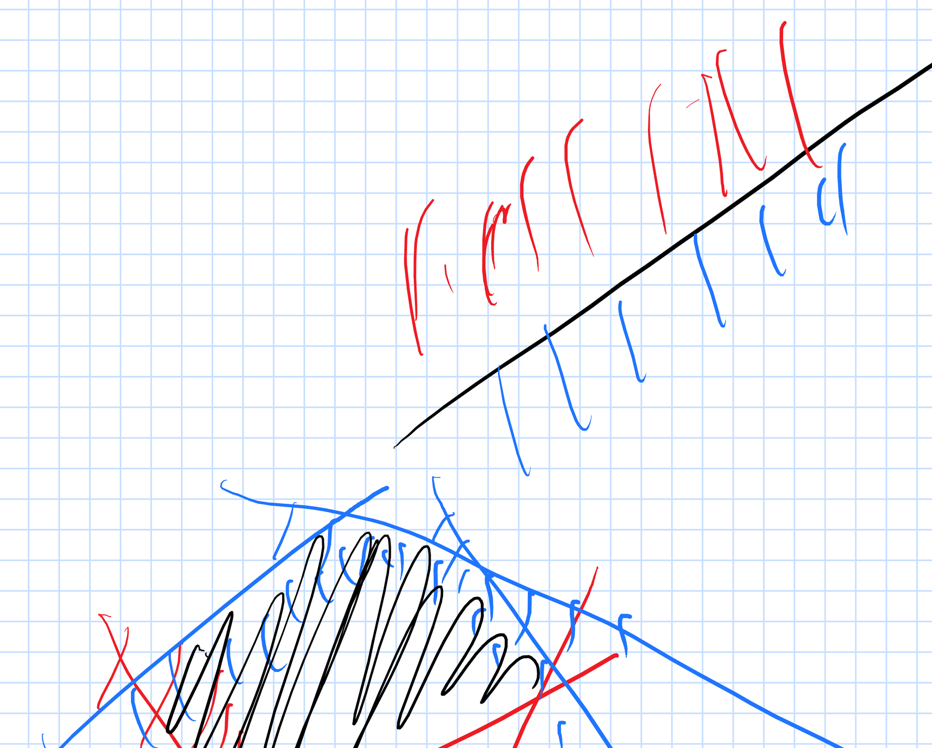
$\sum a_i x_i = b$

$$\boxed{\begin{array}{l} \sum a_i x_i \le b \\ \sum a_i x_i \ge b \end{array}}$$



$\max 7x + 7y = \max \langle (7,7), (x,y) \rangle$

$\sum a_i x + b_i y \le c_i \quad i = 1, \dots, m$



$O(n)$

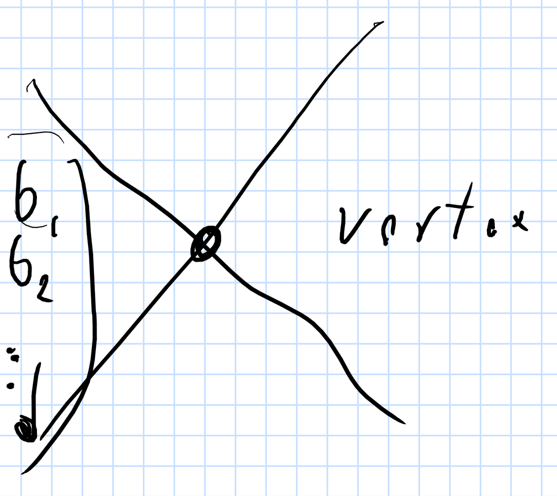
---

system of linear equalities

$\sum a_{i,j} x_i = b_i \quad i = 1, \dots, d$

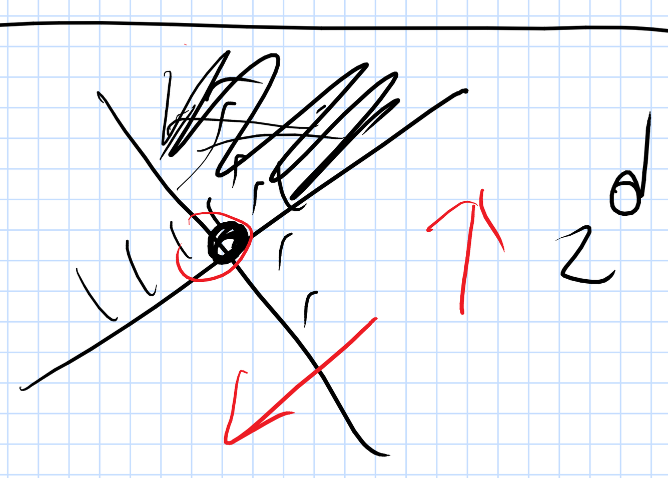
d hyperplane

$$A \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,d} \\ & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix}$$



Graham-Shmit elimination

$O(d^3)$

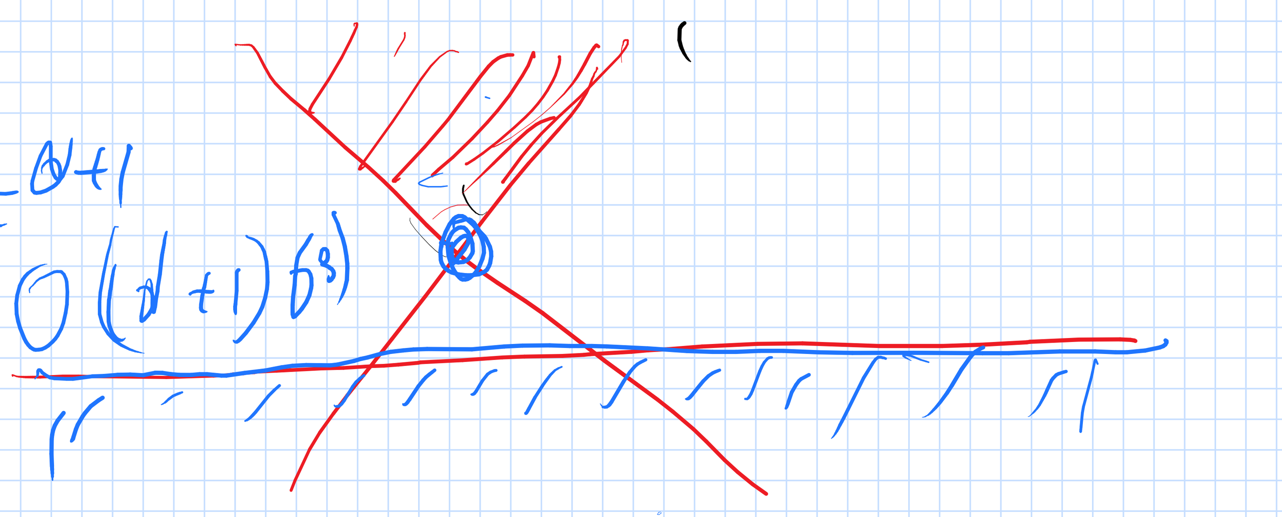


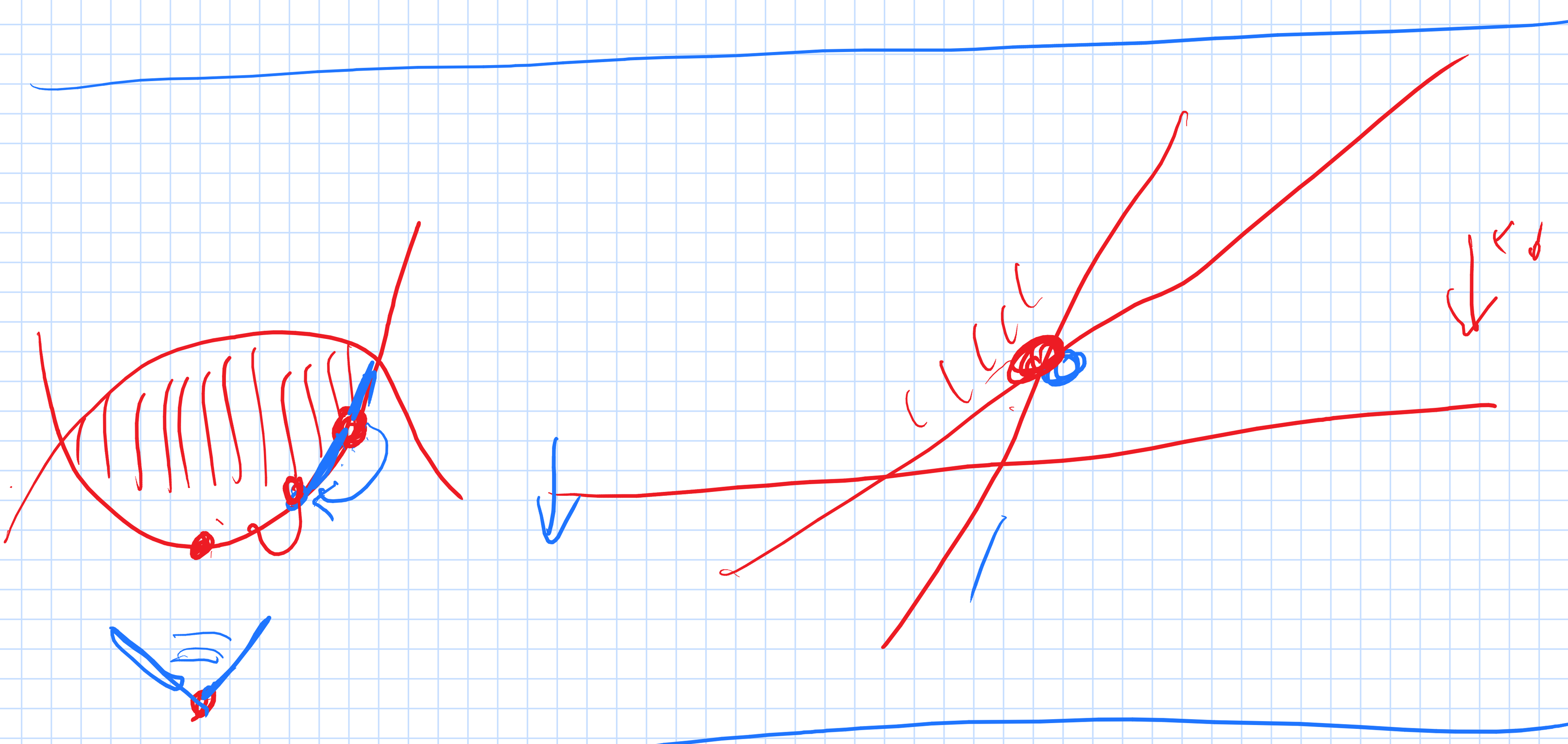
2^d

$d + 1$

$\binom{d+1}{d} = d + 1$

$O((d+1) d^3)$

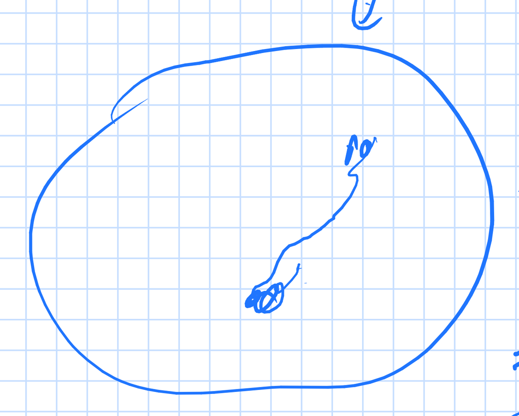




\# of feasible vertices

$\Theta(n^{\lfloor d/2 \rfloor})$

d



diameter of polytope

$\le n + d - 1$

Hirsch conjecture

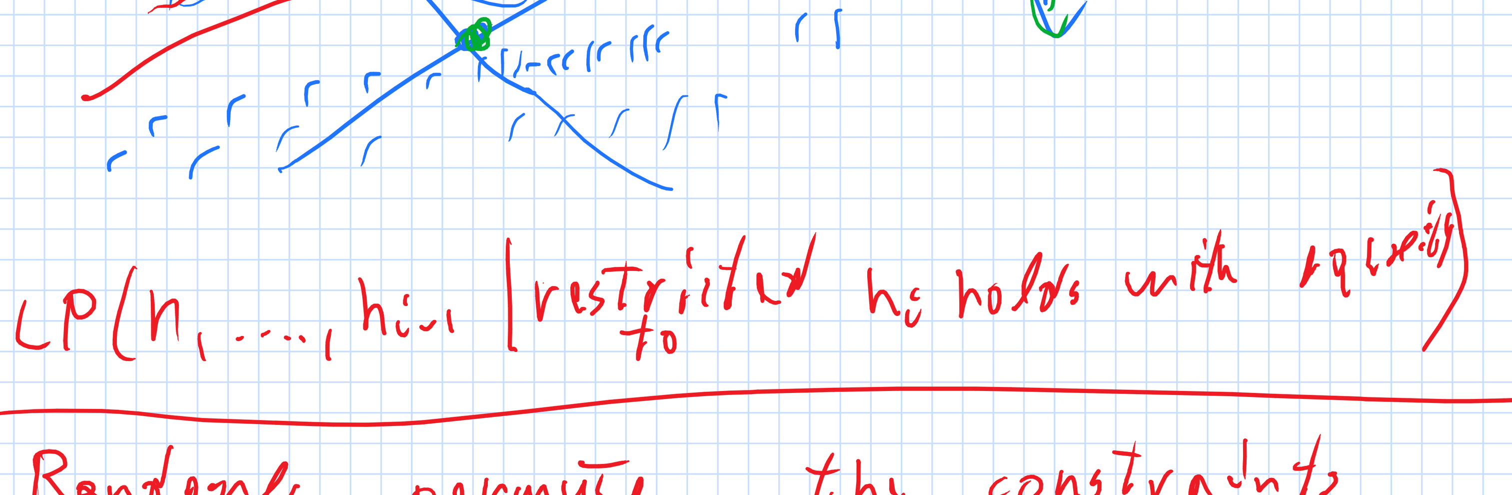
$\ge n + d + 17$

$n^{O(\log d)}$

$h_1, h_2, \dots, h_n$ constraints in d dimensions

$h_1, \dots, h_d$ = define a vertex that is feasible

Incremental construction

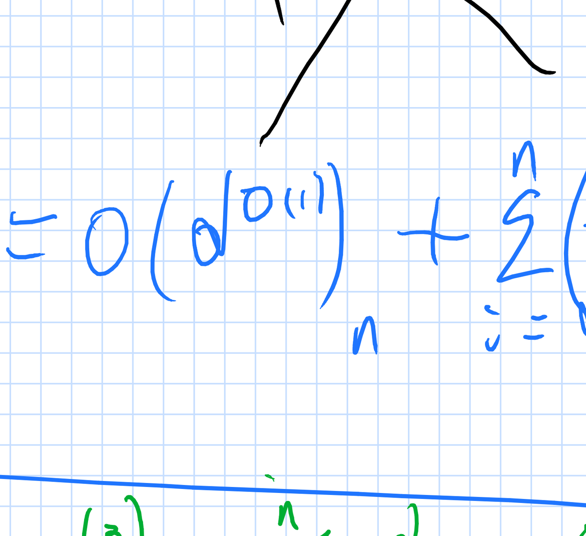


$LP(h_1, \dots, h_{i-1} \mid$ restricted to h_i holds with equality$)$

Randomly permute the constraints

$\underbrace{h_1, \dots h_d}_{\text{basis}}, h_{d+1}, \dots h_n$

Backward analysis



$\frac{d}{i-d}$

$$T(n,d) = O(d^{O(1)}) + \sum_{i=d+1}^{n} \left(\frac{d}{i-d} T(i, d-1) + O(d) \right)$$

$$T(n,d) = O(d^3) + \sum_{i=d}^{n} \frac{d}{i-d} T(i, d-1)$$

$$= O(d^4) + \sum_{i=2d}^{n} \frac{d}{i-d} T(i, d-1)$$

$$= O(d^4) + \sum_{i=2d}^{n} \frac{2d}{i} T(i, d-1)$$

$T(n,d) \le c_d \cdot n$

$$T(n,d) = O(d^4) + \sum_i \frac{2d}{i} c_{d-1} i = O(n)$$