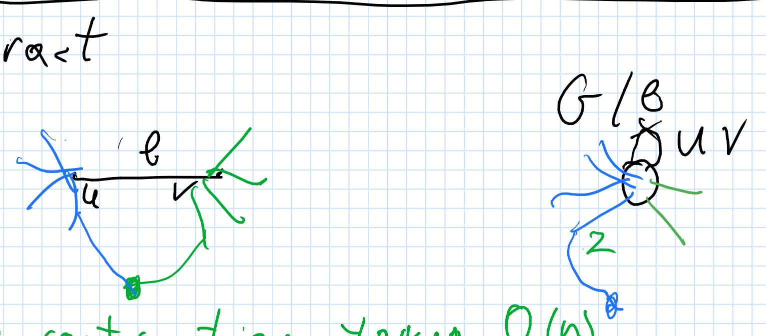
473, L14: Min Cut

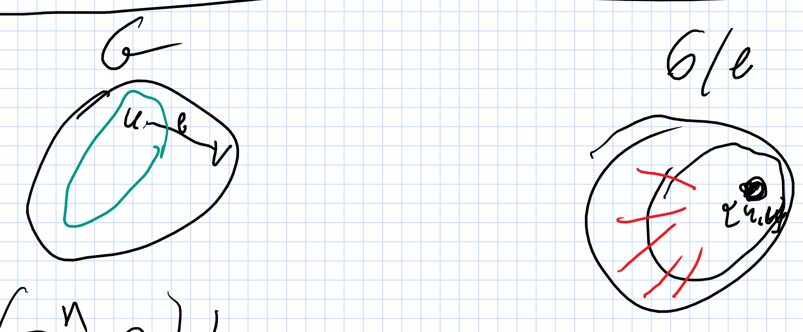
$G=(V,E)$ — n vertices, m edges, undirected, unweighted

$S \subseteq V$ $\quad (S,\bar{S}) = \{uv \in E \mid u \in S \text{ and } v \in V \setminus S\}$

contract



edge contraction takes $O(n)$ time.

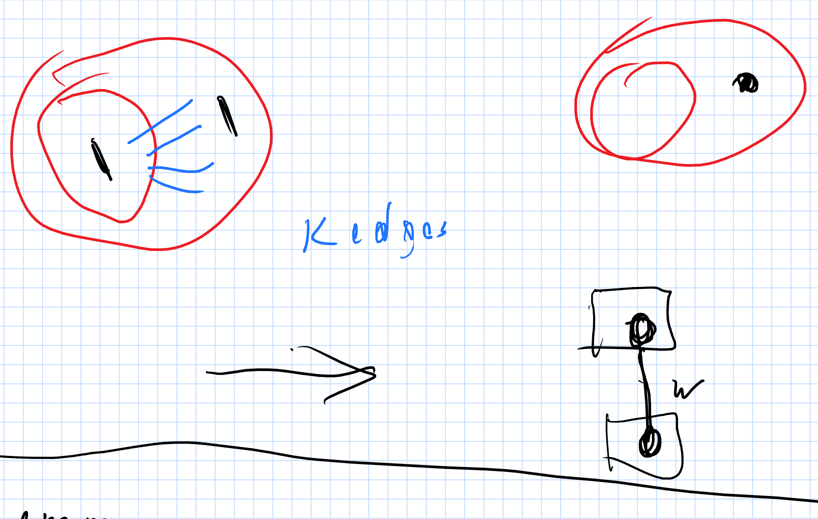


$(2^n-2)/2$

$= 2^{n-1}-1$

$2^{2-2}-1$

e not in the min cut



Lemma
A graph with min cut of size k has $\geq \frac{nk}{2}$ edges.

proof: $|E(G)| = \frac{1}{2}\sum_{v \in V} d(v) \geq \frac{nk}{2}$. ∎

$d(v) \geq k$

$P\left[\begin{matrix}\text{choosing edge}\\ \text{in the cut if graph has } n \text{ vertices}\end{matrix}\right] = \frac{k}{\underbrace{|E(G)|}_{\geq \frac{nk}{2}}} \leq \frac{k}{nk/2} = \frac{2}{n}$.

$P[X \cap Y] = P[X \mid Y]\,P[Y]$

$P\left[\bigcap_{i=1}^{t} E_i\right] = \prod_{i=1}^{t} P[E_i \mid E_1 \cap \dots \cap E_{i-1}]$

$P[E_i \mid E_1 \cap E_2 \cdots \cap E_{i-1}] = P\left[\begin{matrix}\text{pick edge } \underline{\text{not}} \text{ in min cut of size } k\\ \text{given that graph has } n-i+1 \text{ vertices}\end{matrix}\right] \geq 1 - \frac{2}{n-i+1}$

$P\left[\begin{matrix}\text{alg outputs}\\ \text{min cut}\end{matrix}\right] = P[E_1 \cap E_2 \cap \dots \cap E_{n-2}]$

$= \prod_{i=1}^{n-1} P[E_i \mid E_1 \cap E_2 \cap \dots \cap E_{i-1}]$

$\geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{n-i+1}\right)$

$= \prod_{i=1}^{n-2} \frac{n-i+1-2}{n-i+1}$

$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} \cdots \frac{1}{3}$

$= \frac{2 \cdot 1}{n(n-1)} = \frac{2}{n(n-1)}$. ∎

$p = \frac{2}{n(n-1)}$

$O(n^2)$ running time alg outputs min cut with prob $\geq \frac{2}{n(n-1)}$

$O(\frac{1}{p})$ times constant probability

$(1-p)^{1/p} \ll \frac{1}{2}$

$\frac{1}{p} = O(n^2)$

# of times	success prob	RT
1	$\geq 2/n(n-1)$	$O(n^2)$
$O(n^2)$	$\geq \frac{1}{2}$	$O(n^4)$
$O(n^2 \log n)$	$\geq 1-\frac{1}{n}$	$O(n^4 \log n)$

Lemma
Contracting the graph from n vertices to $n/\sqrt{2}$ vertices has probability $\geq \frac{1}{2}$ to succeed.

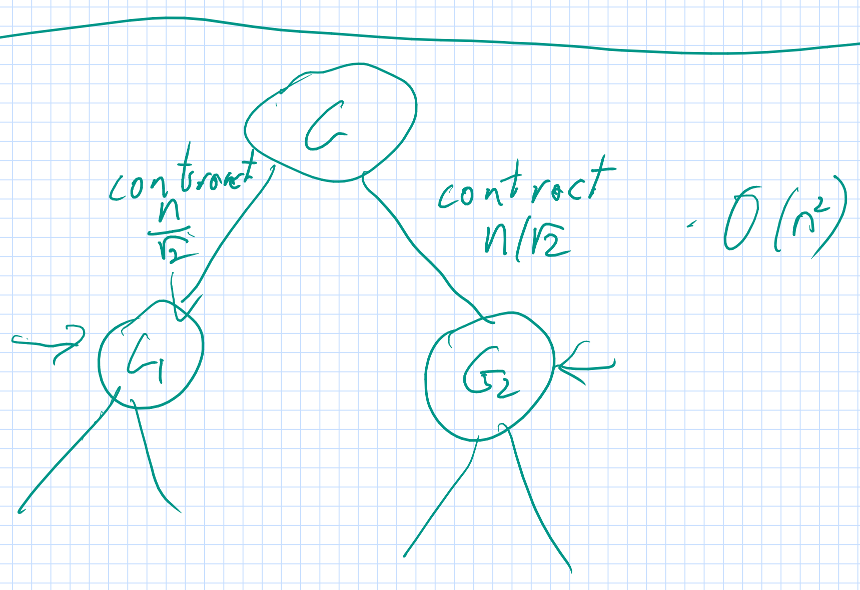
proof

$P\left[\bigcap_{i=n}^{n-n/\sqrt{2}} E_i\right] \geq \prod_{i=1}^{n-n/\sqrt{2}} \left(1 - \frac{2}{n-i+1}\right)$

$= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdots \frac{n/\sqrt{2}-1}{n/\sqrt{2}}$

$\approx \frac{(n/\sqrt{2}-2)(n/\sqrt{2}-1)}{n(n-1)} \geq \frac{1}{2}$

Fast cut

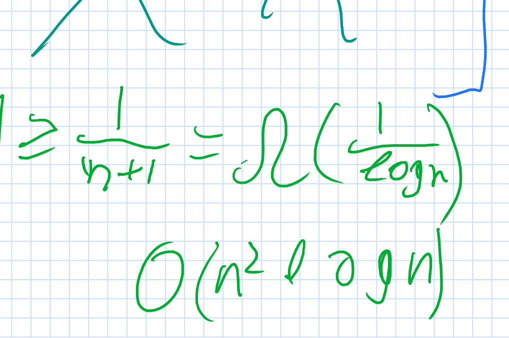


$O(n^2)$

$T(n) = O(n^2) + 2T\left(\frac{n}{\sqrt{2}}\right)$

$= O(n^2 \log n)$

what is the probability of success of Fast cut?



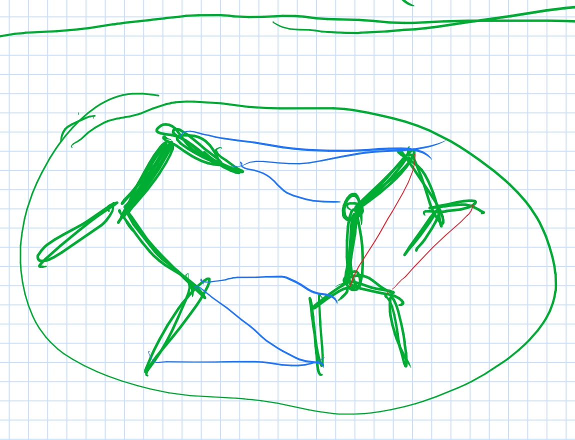
$h = \lceil \log_{\sqrt{2}} n \rceil$

$P[\text{blue path}] \geq \frac{1}{h+1} = \Omega\left(\frac{1}{\log n}\right)$

$p = \Omega\left(\frac{1}{\log n}\right)$ $\qquad O(n^2 \log n)$

# times run		RT
1	$\Omega(\frac{1}{\log n})$	$O(n^2 \log n)$
$\frac{1}{p} = O(\log n)$	$\geq \frac{1}{2}$	$O(n^2 \log^2 n)$
$O(\log^2 n)$	$\geq 1 - \frac{1}{n^{O(1)}}$	$O(n^2 \log^3 n)$

$O(m \log n)$



$O(m \log n)$

$O(m)$ $\qquad \geq \frac{2}{n(n-1)}$