

**31** (100 PTS.) Application: Disjoint Paths

A flow  $f$  is *acyclic* if the subgraph of directed edges with positive flow contains no directed cycle.

- 31.A.** (50 PTS.) Show that given any feasible flow  $f$  in  $G$ , there is a feasible acyclic flow of the same value (This implies that some maximum flow is acyclic).
- 31.B.** (10 PTS.) Give an algorithm to determine a maximum acyclic integer flow in a flow network with integer capacities. What is the running time of your algorithm (Assume  $T(n, m)$  is the running time to find a maximum integer flow in the graph)?
- 31.C.** (40 PTS.) Given any directed graph  $G$  and two vertices  $s$  and  $t$ , output the maximum number of edge disjoint paths from  $s$  to  $t$ . What is the running time of your algorithm?

**32** (100 PTS.) Application-Committee Management

The Computer Science Department at UIUC has  $n$  professors. They handle department duties by taking part in various committees. There are  $m$  committees and the  $j$ th committee requires  $k_j$  professors. The head of the department asked each professor to volunteer for a set of committees. Let  $S_i \subseteq \{1, 2, \dots, m\}$  be the set of committees that professor  $i$  has volunteered for. A committee assignment consists of sets  $S'_1, S'_2, \dots, S'_n$  where  $S'_i \subseteq \{1, 2, \dots, m\}$  is the set of committees that professor  $i$  will participate in. A *valid* committee assignment has to satisfy two constraints: (i) for each professor  $i$ ,  $S'_i \subseteq S_i$ , that is each professor is only given committees that he/she has volunteered for, and (ii) each committee  $j$  has  $k_j$  professors assigned to it, or in other words  $j$  occurs in at least  $k_j$  of the sets  $S'_1, S'_2, \dots, S'_n$ .

- 32.A.** (50 PTS.) Describe a polynomial time algorithm that the head of the department can employ to check if there is a valid committee assignment given  $m, k_1, k_2, \dots, k_m$  the requirements for the committees, and the lists  $S_1, S_2, \dots, S_n$ . The algorithm should output a valid assignment if there is one.
- 32.B.** (50 PTS.) The head of the department notices that often there is no valid committee assignment because professors naturally are inclined to volunteer for as few committees as possible. To overcome this, the definition of a valid assignment is relaxed as follows. Let  $\ell$  be some integer. An assignment  $S'_1, S'_2, \dots, S'_n$  is now said to be valid if (i)  $|S'_i - S_i| \leq \ell$  and (ii) each committee  $j$  has  $k_j$  professors assigned to it. The new condition (i) means that a professor  $i$  may be assigned up to  $\ell$  committees not on the list  $S_i$  that he/she volunteered for. Describe a polynomial time algorithm to check if there is a valid committee assignment with the relaxed definition.

**33** (100 PTS.) More on Disjoint Paths

Let  $G = (V, E)$  be a directed graph and let  $u, v, w$  be distinct vertices. Suppose there are  $k$  edge disjoint paths from  $u$  to  $v$  in  $G$ , and  $k$  edge disjoint paths from  $v$  to  $w$  in  $G$ . Note that the paths from  $u$  to  $v$  can share edges with the paths from  $v$  to  $w$ . Prove that there are  $k$  edge disjoint paths from  $u$  to  $w$  in  $G$ .