All problems are of equal value.

1. Suppose $f$ is an $(s,t)$-flow in a network $G = (V,E)$ and let $f'$ be another $(s,t)$-flow. Prove that there is an $(s,t)$-flow of value $|f'| - |f|$ in the residual network $G_f$, where $|f|, |f'|$ are the values of the flows $f, f'$ respectively. Note that $|f'| - |f|$ may be negative.

Using the above briefly argue that if $f$ is an $(s,t)$-flow then $f$ is a maximum-flow iff there is no $s$-$t$ path in $G_f$. Note that this gives an alternate proof of correctness of the correctness of the augmenting path algorithms.

2. Suppose $G = (V,E)$ is a flow-network with integer capacities and $f$ is an integer valued $s$-$t$ flow in $G$. Let $G' = (V,E')$ be the support of $f$; more formally $E' = \{ e \in E \mid f(e) > 0 \}$.

- Given $G$ and $f$ describe a linear time algorithm to check whether $f$ is a maximum flow in $G$.
- Suppose $e$ is contained in a directed cycle in $G'$. Show that there is another flow $f'$ such that $f'(e) \leq f(e) - 1$ and $|f'| = |f|$.
- Assuming $f$ is a maximum flow in $G$ describe a linear-time algorithm to find a new maximum flow where the capacity of a specific edge $e$ is reduced by one unit.

3. Let $G = (V,E)$ be a flow network with integer edge capacities. We have seen algorithms that compute a minimum $s$-$t$ cut via maximum flow. For the problem below assume that you only have black box access to an algorithm that given $G$ and nodes $s, t$ outputs a minimum cut between $s$ and $t$.

- Describe a simple example of a flow-network $G$ and two nodes $s, t$ such that there are two distinct $s$-$t$ minimum cuts with the same capacity but different number of edges in the cuts.
- Given $G$ and $s, t$ and an integer $k$ describe an algorithm that checks whether $G$ has a minimum cut with at most $k$ edges.

No proof of correctness necessary but we recommend a brief justification. And make sure you have a clear and understandable algorithm.