All problems are of equal value.

1. (Multiplicative Chernoff-Hoeffding Bound). Let $X_1, \ldots, X_n$ be independent random variables taking values over the unit interval $[0, 1]$. Let $X = \sum_i X_i$. Show the following.

(a) For $r \in (-\infty, \ln 2]$, prove that $\mathbb{E}[e^{rX}] \leq e^{r\mathbb{E}[X] + r^2\mathbb{E}[X]}$, where you may use (without proof) that $1 + z \leq e^z$ for all $z$, and that $e^z \leq 1 + z + z^2$ for $z \in (-\infty, \ln 2]$.

(b) Explain how the above used the independence of the $X_i$.

(c) Apply Markov’s inequality ($\Pr[Y \geq a] \leq \mathbb{E}[Y]/a$) to $e^{rX}$, and optimize over $r$, to conclude that:

i. For $0 \leq \epsilon \leq \ln 4$, $\Pr[X \geq (1 + \epsilon)\mathbb{E}[X]] \leq e^{-\epsilon^2\mathbb{E}[X]/4}$.

ii. For $\epsilon \geq \ln 4$, $\Pr[X \geq (1 + \epsilon)\mathbb{E}[X]] \leq 2^{-\epsilon\mathbb{E}[X]/2}$.

iii. For $0 \leq \epsilon \leq 1$, $\Pr[X \leq (1 - \epsilon)\mathbb{E}[X]] \leq e^{-\epsilon^2\mathbb{E}[X]/4}$.

iv. (Additive Chernoff-Hoeffding Bound) For $\epsilon \geq 0$, $\Pr[|X - \mathbb{E}[X]| \geq \epsilon \cdot n] \leq 2e^{-\epsilon^2 n/4}$.

Note: The additive Chernoff-Hoeffding bound suffices for applications such as estimating the errors in polling, but the multiplicative bound is in general stronger and often needed (e.g. consider $\mathbb{E}[X] = \ln n$ and the resulting bound for $\Pr[X \geq 2\mathbb{E}[X]]$). Note also that the above omits one range of parameters, where one can show that $\Pr[X \geq (1 + \epsilon)\mathbb{E}[X]] \leq e^{-(1+\epsilon)\ln(1+\epsilon)\mathbb{E}[X]/4}$ if $\epsilon \geq 1$.

2. For $0 < p \leq 1$, let $\text{Geom}(p)$ denote the geometric distribution with parameter $p$, so that if $X \sim \text{Geom}(p)$ then for $i \geq 1$ we have $\Pr[X = i] = p(1 - p)^{i-1}$. Let $X_1, \ldots, X_n$ be independent random variables distributed according to $\text{Geom}(p)$, and let $X = \sum_{i=1}^n X_i$.

(a) Compute $\mathbb{E}[X]$ and $\text{Var}(X)$.

(b) For $c \geq 2$, use Chebyshev’s inequality to give a bound for $\Pr[X \geq c\mathbb{E}[X]]$.

(c) Let $\text{Bern}(p)$ denote the Bernoulli distribution with parameter $p$, so that if $Y \sim \text{Bern}(p)$ then $\Pr[Y = 1] = p = 1 - \Pr[Y = 0]$. Let $Y_1, \ldots, Y_t$ be independent random variables distributed according to $\text{Bern}(p)$, and let $Y = \sum_{j=1}^t Y_j$. Show that $\Pr[X \geq t] \leq \Pr[Y \leq n]$.

Hint: Consider the following experiment. You are given a biased coin that yields heads with probability $p$, and tails with probability $1 - p$. Flip the coin until you both (a) have seen $n$ heads, and (b) have flipped at least $t$ times in total. Using only this experiment as a probability space, construct random variables identically distributed to the $X_i$ and $Y_j$. First try proving that $\Pr[X > t] = \Pr[Y < n]$.

(d) For $c \geq 2$, use Problem 1 to give a bound for $\Pr[X \geq c\mathbb{E}[X]]$.

3. Consider the following variant of quicksort. Given an array $A$ of $n$ distinct numbers the algorithm picks a pivot $x$ uniformly at random from $A$ and computes the rank of $x$. If the rank of $x$ is between $n/4$ and $3n/4$ (a balanced pivot), the algorithm proceeds as with standard quicksort. However, if the pivot is not balanced, the algorithm selects a new random pivot.
(a) Write a formal description of the algorithm.
(b) Directly prove that the expected runtime of this algorithm is $O(n \log n)$.
(c) For $c \geq 2$, bound the probability that the algorithm takes more than $c \cdot \Theta(n \log n)$ steps.
   
   Hint: Use Problem 2.

(d) For a random variable $Z$ over non-negative integers $\mathbb{N}$, prove that $\mathbb{E}[Z] = \sum_{i \geq 1} \Pr[Z \geq i]$.
   
   Hint: Use indicator random variables, and linearity of expectation over an infinite sum.
   (There are non-trivial issues of convergence here; ignore them.)

(e) Use the previous two parts to reprove that the expected runtime is $O(n \log n)$. 