Universal and Perfect Hashing

Lecture 10

September 26, 2019

Announcements and Overview

- Pset 4 released and due on Thursday, October 3 at 10am. Note one day extension over usual deadline.
- Midterm 1 is on Monday, Oct 7th from 7-9.30pm. More details and conflict exam information will be posted on Piazza.
- Next pset will be released after the midterm exam.

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Today's lecture:

- Review pairwise independence and related constructions
- (Strongly) Universal hashing
- Perfect hashing

Part I

Review

Pairwise independent random variables

Definition

Random variables $X_1, X_2, \ldots, X_n$ from a range B are **pairwise** independent if for all $1 \leq i < j \leq n$ and for all $b, b' \in B$,

$$\Pr[X_i = b, X_j = b'] = \Pr[X_i = b] \cdot \Pr[X_j = b'] .$$

Constructing pairwise independent rvs

Suppose we want to create n pairwise independent random variables in range $0, 1, \ldots, m-1$. That is we want to generate $X_0, X_2, \ldots, X_{n-1}$ such that

- $\Pr[X_i = \alpha] = 1/m$ for each $\alpha \in \{0, 1, 2, \ldots, m-1\}$
- X_i and X_j are independent for any $i \neq j$

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Interesting case: $n = m = p$ where p is a prime number

- Pick a, b uniformly at random from $\{0, 1, 2, \ldots, p-1\}$
- Set $X_i = ai + b$
- Only need to store a, b. Can generate X_i from i.

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- Only need to store a, b. Can generate X_i from i.

Relies on the fact that $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ is a field

Pairwise independence for general n and m

A rough sketch.
If $n < m$ we can use a prime $p \in [m, 2m]$ (one always exists) and use the previous construction based on $\mathbb{Z}_p$.

$n > m$ is the more difficult case and also relevant.

The following is a fundamental theorem on finite fields.

Theorem

Every finite field $\mathbb{F}$ has order p^k for some prime p and some integer $k \geq 1$. For every prime p and integer $k \geq 1$ there is a finite field $\mathbb{F}$ of order p^k and is unique up to isomorphism.

We will assume n and m are powers of 2. From above can assume we have a field $\mathbb{F}$ of size $n = 2^k$.

Pairwise independence when n, m are powers of 2

We will assume n and m are powers of 2.
We have a field $\mathbb{F}$ of size $n = 2^k$.

Generate n pairwise independent random variables from $[n]$ to $[n]$ by picking random $a, b \in \mathbb{F}$ and setting $X_i = ai + b$ (operations in $\mathbb{F}$). From previous proof $X_1, \ldots, X_n$ are pairwise independent.

Now $X_i \in [n]$. Truncate X_i to $[m]$ by dropping the most significant $\log n - \log m$ bits. Resulting variables are still pairwise independent (both n, m being powers of 2 important here).

Skipping details on computational aspects of $\mathbb{F}$ which are closely tied to the proof of the theorem on fields.

Pairwise Independence and Chebyshev's Inequality

Chebyshev's Inequality

For $a \geq 0$, $\Pr[|X - \mathbf{E}[X]| \geq a] \leq \frac{Var(X)}{a^2}$ equivalently for any $t > 0$, $\Pr[|X - \mathbf{E}[X]| \geq t\sigma_X] \leq \frac{1}{t^2}$ where $\sigma_X = \sqrt{Var(X)}$ is the standard deviation of X.

Suppose $X = X_1 + X_2 + \ldots + X_n$.
If $X_1, X_2, \ldots, X_n$ are independent then $Var(X) = \sum_i Var(X_i)$.

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Suppose $X = X_1 + X_2 + \ldots + X_n$.
If $X_1, X_2, \ldots, X_n$ are independent then $Var(X) = \sum_i Var(X_i)$.

Lemma

Suppose $X = \sum_i X_i$ and $X_1, X_2, \ldots, X_n$ are pairwise independent, then $Var(X) = \sum_i Var(X_i)$.

Hence pairwise independence suffices if one relies only on Chebyshev inequality.

Part II

Hash Tables

Dictionary Data Structure

1. $\mathcal{U}$: universe of keys with total order: numbers, strings, etc.
2. Data structure to store a subset $S \subseteq \mathcal{U}$
3. **Operations:**
   1. **Search/look up**: given $x \in \mathcal{U}$ is $x \in S$?
   2. **Insert**: given $x \notin S$ add x to S.
   3. **Delete**: given $x \in S$ delete x from S
4. **Static** structure: S given in advance or changes very infrequently, main operations are lookups.
5. **Dynamic** structure: S changes rapidly so inserts and deletes as important as lookups.

Can we do everything in $O(1)$ time?

Hashing and Hash Tables

Hash Table data structure:

1. A (hash) table/array T of size m (the table **size**).
2. A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$.
3. Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T.

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2. A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$.
3. Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in T.

Given $S \subseteq \mathcal{U}$. How do we store S and how do we do lookups?

Ideal situation:

1. Each element $x \in S$ hashes to a distinct slot in T. Store x in slot $h(x)$
2. **Lookup**: Given $y \in \mathcal{U}$ check if $T[h(y)] = y$. $O(1)$ time!

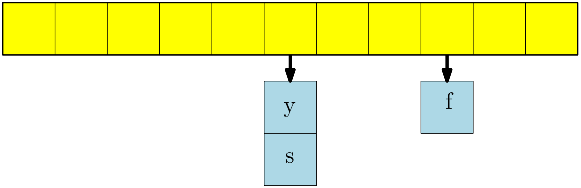
Collisions unavoidable if $|T| < |\mathcal{U}|$.

Handling Collisions: Chaining

Collision: $h(x) = h(y)$ for some $x \neq y$.

Chaining/Open hashing to handle collisions:

1. For each slot i store all items hashed to slot i in a linked list. $T[i]$ points to the linked list
2. **Lookup**: to find if $y \in \mathcal{U}$ is in T, check the linked list at $T[h(y)]$. Time proportion to size of linked list.



Does hashing give $O(1)$ time per operation for dictionaries?

Hash Functions

Parameters: $N = |\mathcal{U}|$ (very large), $m = |T|$, $n = |S|$
Goal: $O(1)$-time lookup, insertion, deletion.

Single hash function

If $N \geq m^2$, then for any hash function $h : \mathcal{U} \to T$ there exists $i < m$ such that at least $N/m \geq m$ elements of $\mathcal{U}$ get hashed to slot i. Any S containing all of these is a **very very bad set for h**!

Such a bad set may lead to $O(m)$ lookup time!

In practice:

- Dictionary applications: choose a simple hash function and hope that worst-case bad sets do not arise
- Crypto applications: create "hard" and "complex" function very carefully which makes finding collisions difficult

Hashing from a theoretical point of view

- Consider a family $\mathcal{H}$ of hash functions with *good properties* and choose h randomly from $\mathcal{H}$
- Guarantees: small # collisions in expectation for any given S.
- $\mathcal{H}$ should allow efficient sampling.
- Each $h \in \mathcal{H}$ should be efficient to evaluate and require small memory to store.

In other words a hash function is a "pseudorandom" function

Strongly Universal Hashing

1. **Uniform:** Consider any element $x \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then x should go into a random slot in T. In other words $\Pr[h(x) = i] = 1/m$ for every $0 \le i < m$.
2. **(2)-Strongly Universal**: Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then $h(x)$ and $h(y)$ should be independent random variables.

Universal Hashing

- **(2)-Universal:** Consider any two distinct elements $x, y \in \mathcal{U}$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between x and y should be at most $1/m$. In other words $\Pr[h(x) = h(y)] \leq 1/m$.

Note: we do not insist on uniformity.

Universal Hashing

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Note: we do not insist on uniformity.

Universal hashing is a relaxation of strong universal hashing and simpler to construct while retaining most of the useful properties.

(Strongly) Universal Hashing

Definition

A family of hash functions $\mathcal{H}$ is (2-)**strongly universal** if for all distinct $x, y \in \mathcal{U}$, $h(x)$ and $h(y)$ are independent for h chosen uniformly at random from $\mathcal{H}$, and for all x, $h(x)$ is uniformly distributed.

Definition

A family of hash functions $\mathcal{H}$ is (2-)**universal** if for all distinct $x, y \in \mathcal{U}$, $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq 1/m$ where m is the table size.

Analyzing Universal Hashing

1. T is hash table of size m.
2. $S \subseteq \mathcal{U}$ is a **fixed** set of size n
3. h is chosen randomly from a universal hash family $\mathcal{H}$.
4. x is a *fixed* element of $\mathcal{U}$.

Question: What is the *expected* time to look up x in T using h assuming chaining used to resolve collisions?

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1. The time to look up x is the size of the list at $T[h(x)]$: same as the number of elements in S that collide with x under h.
2. $\ell(x)$ be this number. We want $\mathrm{E}[\ell(x)]$
3. Let $C_{x,y}$ be indicator random variable for x, y colloding under h, that $C_{x,y} = 1$ iff $h(x) = h(y)$

Analyzing Universal Hashing

Continued...

Number of elements colliding with x: $\ell(x) = \sum_{y \in S} C_{x,y}$.

$$
\begin{aligned}
\Rightarrow \mathrm{E}[\ell(x)] &= \sum_{y \in S, y \neq x} \mathrm{E}[C_{x,y}] && \text{linearity of expectation} \\
&= \sum_{y \in S, y \neq x} Pr[h(x) = h(y)] \\
&\leq \sum_{y \in S, y \neq x} \frac{1}{m} && (\text{since } \mathcal{H} \text{ is a universal hash family}) \\
&\leq |S|/m \\
&\leq \frac{n}{m} \\
&\leq 1 && (\text{if } |S| \leq m)
\end{aligned}
$$

Analyzing Universal Hashing

Comments:

1. Expected time for insertion and deletion also $O(1)$ if $n \leq m$.
2. Analysis assumes static set S but holds as long as S is a set formed with at most $O(m)$ insertions and deletions. Assumption is that insertions and deletions are not adaptive.
3. **Worst-case**: look up time can be large! How large? Technically $O(n)$ if all elements collide.

Analyzing Universal Hashing: Maximum Load

If h is a fully random function and $m = n$ then expected maximum load in any bucket of T is $O(\log n / \log\log n)$ via balls and bin analogy.

If h is chosen from a universal hash family $\mathcal{H}$ what is the expected maximum load?

Lemma

Let h be chosen from a universal hash family and let $m \geq n$ and let L be maximum load of any slot. Then $\Pr\left[L > t\sqrt{n}\right] \leq 1/t^2$ for $t \geq 1$.

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Thus $L = O(\sqrt{n})$ with probability at least $1/2$.

Analyzing Universal Hashing: Maximum Load

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Let $C = \sum_{x,y \in S, x \neq y} C_{x,y}$ be total number of collisions.

- $\mathbf{E}[C] \leq \binom{n}{2}/m \leq (n-1)/2$ if $m \geq n$.
- **Observation:** $C \geq \binom{L}{2}$. Why?
- $L > t\sqrt{n}$ implies $C > t^2 n/2$.
- By Markov $\Pr[C > t^2 n/2] \leq \mathbf{E}[C]/(t^2 n/2) \leq 1/t^2$
- Hence $\Pr[L > t\sqrt{n}] \leq 1/t^2$.

Analyzing Universal Hashing: Maximum Load

Lemma

Let h be chosen from a universal hash family and let $m \geq n$ and let L be maximum load of any slot. Then $\mathbf{E}[L] = O(\sqrt{n})$.

Direct proof: $(\mathbf{E}[L])^2 \leq \mathbf{E}\left[L^2\right] \leq \mathbf{E}[C] \leq n$ (using Jensen's ineq)

L is a non-negative random variable in range. Hence

$$
\begin{aligned}
E[L] &= \sum_{i=1}^{n} \Pr[L \geq i] \quad \text{(from defn of expectation)} \\
&\leq \sum_{i=1}^{\sqrt{n}} 1 + \sum_{i=\sqrt{n}+1}^{n} n/i^2 \quad \text{(from previous lemma)} \\
&\leq \sqrt{n} + n \int_{\sqrt{n}}^{n} 1/i^2 \leq 2\sqrt{n}.
\end{aligned}
$$

Compact Strongly Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

Question: How do we construct strongly universal hash family?

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If N and m are powers of 2 then use construction of N pairwise independent random variables over range $[m]$ discussed previously

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Question: How do we construct strongly universal hash family?

If N and m are powers of 2 then use construction of N pairwise independent random variables over range $[m]$ discussed previously

Disadvantage: Need m to be power of 2 and requires complicated field operations

Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

1. Choose a **prime** number $p > N$. Define function $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
2. Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$ ($\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$). Note that $|\mathcal{H}| = p(p - 1)$.

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Theorem

$\mathcal{H}$ is a universal hash family.

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Theorem

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Comments:

1. $h_{a,b}$ can be evaluated in $O(1)$ time.
2. Easy to store, *i.e.*, just store a, b. Easy to sample.

Understanding the hashing

Once we fix a and b, and we are given a value x, we compute the hash value of x in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

Let $g_{a,b}(x) = (ax + b) \mod p$.
$h_{a,b}(x) = g_{a,b}(x) \mod m$.

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Fix x:

- $g_{a,b}(x)$ is uniformly distributed in $\{0, 1, \ldots, p - 1\}$. Why?
- However $h_{a,b}(x)$ is not necessarily uniformly distributed over $\{0, 1, 2, \ldots, m\}$. Why?

Some math required...

Recall $\mathbb{Z}_p$ is a field.

- $a \neq 0$ implies unique a' such that $aa' = 1 \mod p$
- For $a, x, y \in \mathbb{Z}_p$ such that $x \neq y$ and $a \neq 0$ we have $ax \neq ay \mod p$.
- For $x \neq y$ and any r, s there is a unique solution (a, b) to the equations $ax + b = r$ and $ay + b = s$.

Proof of the Theorem: Outline

$h_{a,b}(x) = ((ax + b) \mod p) \mod m)$.

Theorem

$\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$ *is universal.*

Proof.

Fix $x, y \in \mathcal{U}, x \neq y$. Show that
$\Pr_{h_{a,b} \sim \mathcal{H}}[h_{a,b}(x) = h_{a,b}(y)] \leq 1/m$.
Note that $|\mathcal{H}| = p(p-1)$.

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1. Let (a, b) (equivalently $h_{a,b}$) be *bad* for x, y if $h_{a,b}(x) = h_{a,b}(y)$.

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1. Let (a, b) (equivalently $h_{a,b}$) be *bad* for x, y if $h_{a,b}(x) = h_{a,b}(y)$.
2. **Claim:** Number of bad (a, b) is at most $p(p-1)/m$.

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1. Let (a, b) (equivalently $h_{a,b}$) be *bad* for x, y if $h_{a,b}(x) = h_{a,b}(y)$.
2. **Claim:** Number of bad (a, b) is at most $p(p-1)/m$.
3. Total number of hash functions is $p(p-1)$ and hence probability of a collision is $\leq 1/m$.

□

Proof of Claim

$$h_{a,b}(x) = (((ax + b) \mod p) \mod m)$$

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b)\ mod\ p$ and $s = (ay + b)\ mod\ p$.

Proof of Claim

$$h_{a,b}(x) = (((ax + b) \mod p) \mod m)$$

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b)\ mod\ p$ and $s = (ay + b)\ mod\ p$.

1. 1-to-1 correspondence between $p(p-1)$ pairs of (a, b) (equivalently $h_{a,b}$) and $p(p-1)$ pairs of (r, s).

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$$h_{a,b}(x) = (((ax + b) \mod p) \mod m)$$

2 lemmas ...

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1. 1-to-1 correspondence between $p(p-1)$ pairs of (a, b) (equivalently $h_{a,b}$) and $p(p-1)$ pairs of (r, s).

2. Out of all possible $p(p-1)$ pairs of (r, s), at most $p(p-1)/m$ fraction satisfies $r\ mod\ m = s\ mod\ m$.

Correspondence Lemma

Lemma

If $x \neq y$ *then for each* (r, s) *such that* $r \neq s$ *and* $0 \leq r, s \leq p-1$ *there is exactly* **one** *pair* (a, b) *such that* $a \neq 0$ *and* $ax + b \mod p = r$ *and* $ay + b \mod p = s$.

Proof.

Solve the two equations:

$$ax + b = r \mod p \qquad \text{and} \qquad ay + b = s \mod p$$

We get $a = \frac{r-s}{x-y} \mod p$ and $b = r - ax \mod p$. □

One-to-one correspondence between (a, b) and (r, s)

Collisions due to folding

Once we fix a and b, and we are given a value x, we compute the hash value of x in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

Collision...

Given two distinct values x and y they might collide only because of folding.

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Collision...

Given two distinct values x and y they might collide only because of folding.

Lemma

$\#$ of pairs (r, s) of $\mathbb{Z}_p \times \mathbb{Z}_p$ such that $r \neq s$ and $r \mod m = s \mod m$ is at most $p(p-1)/m$.

Folding numbers

Lemma

 # pairs $(r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *such that* $r \neq s$ *and* $r \mod m = s \mod m$ *(folded to the same number) is* $p(p-1)/m$.

Proof.

Consider a pair $(r, s) \in \{0, 1, \ldots, p-1\}^2$ s.t. $r \neq s$. Fix r:

1. Let $d = r \mod m$.

Folding numbers

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 # pairs $(r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *such that* $r \neq s$ *and* $r \mod m = s \mod m$ *(folded to the same number) is* $p(p-1)/m$.

Proof.

Consider a pair $(r, s) \in \{0, 1, \ldots, p-1\}^2$ s.t. $r \neq s$. Fix r:

1. Let $d = r \mod m$.
2. There are $\lceil p/m \rceil$ values of s such that $r \mod m = s \mod m$.
3. One of them is when $r = s$.
4. $\implies$ # of colliding pairs

Folding numbers

Lemma

$\#$ pairs $(r,s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ such that $r \neq s$ and $r \mod m = s \mod m$ (folded to the same number) is $p(p-1)/m$.

Proof.

Consider a pair $(r,s) \in \{0,1,\ldots,p-1\}^2$ s.t. $r \neq s$. Fix r:

1. Let $d = r \mod m$.
2. There are $\lceil p/m \rceil$ values of s such that $r \mod m = s \mod m$.
3. One of them is when $r = s$.
4. $\implies$ $\#$ of colliding pairs $(\lceil p/m \rceil - 1)p \leq (p-1)p/m$

□

Proof of Claim

of bad pairs is $\mathbf{p(p-1)/m}$

Proof.

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

1. Let $r = ax + b \mod p$ and $s = ay + b \mod p$.
2. Collision if and only if $r \mod m = s \mod m$.
3. (Folding error): Number of pairs (r, s) such that $r \neq s$ and $0 \leq r, s \leq p - 1$ and $r \mod m = s \mod m$ is $p(p-1)/m$.
4. From previous lemma there is one-to-one correspondence between (a, b) and (r, s). Hence total number of bad (a, b) pairs is $p(p-1)/m$.

□

Proof of Claim

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Prob of x and y to collide: $\frac{\#\text{ bad } (a,b) \text{ pairs}}{\#(a,b) \text{ pairs}} = \frac{p(p-1)/m}{p(p-1)} = \frac{1}{m}$.

Part III

Perfect Hashing

Perfect Hashing

Question: Suppose we get a set $S \subset \mathcal{U}$ of size n. Can we design an "efficient" and "perfect" hash function?

- Create a table T of size $m = O(n)$.
- Create a hash function $h : S \to [m]$ with no collisions!
- h should be fast and efficient to evaluate
- Construct h efficiently given S. Construction of h can be randomized (Las Vegas algorithm)

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A perfect hash function would guarantee lookup time of $O(1)$.

Perfect Hashing via Large Space

Suppose $m = n^2$. Table size is much bigger than n

Lemma

Suppose $\mathcal{H}$ is a universal hash family and $m = n^2$. Then $\Pr_{h \in \mathcal{H}}[\textit{no collisions in } S] \geq 1/2$.

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- $\mathbf{E}[C] \leq \binom{n}{2}/m < 1/2$.
- By Markov inequality $\Pr[C \geq 1] < 1/2$.

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Algorithm: pick $h \in \mathcal{H}$ randomly and check if h is perfect. Repeat until success.

Perfect Hashing

Two levels of hash tables

Question: Can we obtain perfect hashing with $m = O(n)$?

Perfect Hashing

- Do hashing once with table T of size m
- For each slot i in T let Y_i be number of elements hashed to slot i. If $Y_i > 1$ use perfect hashing with second table T_i of size Y_i^2.

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$$Z = m + \sum_{i=0}^{m-1} Y_i^2$$

a random variable (depends on random choice of first level hash function)

Perfect Hashing
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$\mathbf{E}\left[\sum_{i=0}^{m-1} Y_i^2\right] \leq 3n/2$ *if* $m \geq n$.

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Proof.

Let C be total number of collisions. We already saw $\mathbf{E}[C] \leq \binom{n}{2}/m$.
$\sum_i \binom{Y_i}{2} = C$ and hence $\sum_i Y_i^2 = 2C + \sum_i Y_i$.
Therefore

$$\mathbf{E}\left[\sum_i Y_i^2\right] \leq 2\binom{n}{2}/m + \mathbf{E}\left[\sum_i Y_i\right] = 2\binom{n}{2}/m + n \leq 3n/2.$$

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Perfect Hashing

- Do hashing once with table T of size m
- For each slot i in T let Y_i be number of elements hashed to slot i. If $Y_i > 1$ use perfect hashing with second table T_i of size Y_i^2.

Space usage is $Z = m + \sum_{i=0}^{m-1} Y_i^2$ and $\mathbf{E}[Z] \leq 5n/2$ if $m = n$.

- Use algorithm to create perfect hash table
- By Markov space usage is $< 5n$ with probability at least $1/2$
- Repeat if space usage is larger than $5n$. Expected number of repetitions is 2. Hence it leads to $O(n)$ time Las Vegas algorithm
- Technically also need to count the space to store multiple hash functions: $O(n)$ overhead

Rehashing, amortization and...

... making the hash table dynamic

So far we assumed fixed S of size $\simeq m$.

Question: What happens as items are inserted and deleted?

1. If $|S|$ grows to more than cm for some constant c then hash table performance clearly degrades.
2. If $|S|$ stays around $\simeq m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

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Solution: Rebuild hash table periodically!

1. Choose a new table size based on current number of elements in the table.
2. Choose a new random hash function and rehash the elements.
3. Discard old table and hash function.

Question: When to rebuild? How expensive?

Rebuilding the hash table

1. Start with table size m where m is some estimate of $|S|$ (can be some large constant).
2. If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.

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The **amortize** cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when S changes. Hence $O(1)$ expected look up/insert/delete time *dynamic* data dictionary data structure!

Practical Issues

Hashing used typically for integers, vectors, strings etc.

- Universal hashing is defined for integers. To implement for other objects need to map objects in some fashion to integers (via representation)
- Practical methods for various important cases such as vectors, strings are studied extensively. See http://en.wikipedia.org/wiki/Universal_hashing for some pointers.
- Details on Cuckoo hashing and its advantage over chaining http://en.wikipedia.org/wiki/Cuckoo_hashing.
- Relatively recent important paper bridging theory and practice of hashing. "The power of simple tabulation hashing" by Mikkel Thorup and Mihai Patrascu, 2011. See http://en.wikipedia.org/wiki/Tabulation_hashing

Part IV

Bloom Filters

Bloom Filters

Hashing:

1. To insert x in dictionary store x in table in location $h(x)$
2. To lookup y in dictionary check contents of location $h(y)$
3. Storing items in dictionary expensive in terms of memory, especially if items are unwieldy objects such a long strings, images, etc with *non-uniform* sizes.

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Bloom Filter: tradeoff space for false positives

1. To insert x in dictionary set *bit* to $\mathbf{1}$ in location $h(x)$ (initially all bits are set to $\mathbf{0}$)
2. To lookup y if bit in location $h(y)$ is $\mathbf{1}$ say yes, else no.

Bloom Filters

Bloom Filter: tradeoff space for false positives

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Reducing false positives:

1. Pick k hash functions $h_1, h_2, \ldots, h_k$ *independently*

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2. To insert set $h_i(x)$th bit to one in table i for each $1 \leq i \leq k$

Bloom Filters

Bloom Filter: tradeoff space for false positives

1. To insert x in dictionary set *bit* to 1 in location $h(x)$ (initially all bits are set to 0)
2. To lookup y if bit in location $h(y)$ is 1 say yes, else no
3. No false negatives but false positives possible due to collisions

Reducing false positives:

1. Pick k hash functions $h_1, h_2, \ldots, h_k$ *independently*
2. To insert set $h_i(x)$th bit to one in table i for each $1 \leq i \leq k$
3. To lookup y compute $h_i(y)$ for $1 \leq i \leq k$ and say yes only if each bit in the corresponding location is 1, otherwise say no. If probability of false positive for one hash function is $\alpha < 1$ then with k independent hash function it is α^k.