Universal and Perfect Hashing

Lecture 10
September 26, 2019
Announcements and Overview

- Pset 4 released and due on Thursday, October 3 at 10am. Note one day extension over usual deadline.
- Midterm 1 is on Monday, Oct 7th from 7-9.30pm. More details and conflict exam information will be posted on Piazza.
- Next pset will be released after the midterm exam.
Announcements and Overview

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Today’s lecture:
- Review pairwise independence and related constructions
- (Strongly) Universal hashing
- Perfect hashing
Part I

Review
Pairwise independent random variables

**Definition**

Random variables $X_1, X_2, \ldots, X_n$ from a range $B$ are pairwise independent if for all $1 \leq i < j \leq n$ and for all $b, b' \in B$,

$$
\Pr[X_i = b, X_j = b'] = \Pr[X_i = b] \cdot \Pr[X_j = b'].
$$
Suppose we want to create \( n \) pairwise independent random variables in range \( 0, 1, \ldots, m - 1 \). That is we want to generate \( X_0, X_2, \ldots, X_{n-1} \) such that

- \( \Pr[X_i = \alpha] = 1/m \) for each \( \alpha \in \{0, 1, 2, \ldots, m - 1\} \)
- \( X_i \) and \( X_j \) are independent for any \( i \neq j \)
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- $\Pr[X_i = \alpha] = 1/m$ for each $\alpha \in \{0, 1, 2, \ldots, m - 1\}$
- $X_i$ and $X_j$ are independent for any $i \neq j$

Interesting case: $n = m = p$ where $p$ is a prime number

- Pick $a, b$ uniformly at random from $\{0, 1, 2, \ldots, p - 1\}$
- Set $X_i = ai + b$
- Only need to store $a, b$. Can generate $X_i$ from $i$. 
Constructing pairwise independent rvs

Suppose we want to create \( n \) pairwise independent random variables in range \( 0, 1, \ldots, m - 1 \). That is we want to generate \( X_0, X_2, \ldots, X_{n-1} \) such that

- \( \Pr[X_i = \alpha] = 1/m \) for each \( \alpha \in \{0, 1, 2, \ldots, m - 1\} \)
- \( X_i \) and \( X_j \) are independent for any \( i \neq j \)

Interesting case: \( n = m = p \) where \( p \) is a prime number

- Pick \( a, b \) uniformly at random from \( \{0, 1, 2, \ldots, p - 1\} \)
- Set \( X_i = ai + b \)
- Only need to store \( a, b \). Can generate \( X_i \) from \( i \).

Relies on the fact that \( \mathbb{Z}_p = \{0, 1, 2, \ldots, p - 1\} \) is a field
Pairwise independence for general \( n \) and \( m \)

A rough sketch.

If \( n < m \) we can use a prime \( p \in [m, 2m] \) (one always exists) and use the previous construction based on \( \mathbb{Z}_p \).

\( n > m \) is the more difficult case and also relevant.

The following is a fundamental theorem on finite fields.

**Theorem**

Every finite field \( \mathbb{F} \) has order \( p^k \) for some prime \( p \) and some integer \( k \geq 1 \). For every prime \( p \) and integer \( k \geq 1 \) there is a finite field \( \mathbb{F} \) of order \( p^k \) and is unique up to isomorphism.

We will assume \( n \) and \( m \) are powers of 2. From above can assume we have a field \( \mathbb{F} \) of size \( n = 2^k \).
Pairwise independence when \( n, \ m \) are powers of 2

We will assume \( n \) and \( m \) are powers of 2. We have a field \( \mathbb{F} \) of size \( n = 2^k \).

Generate \( n \) pairwise independent random variables from \([n]\) to \([n]\) by picking random \( a, b \in \mathbb{F} \) and setting \( X_i = ai + b \) (operations in \( \mathbb{F} \)). From previous proof \( X_1, \ldots, X_n \) are pairwise independent.

Now \( X_i \in [n] \). Truncate \( X_i \) to \([m]\) by dropping the most significant \( \log n - \log m \) bits. Resulting variables are still pairwise independent (both \( n, m \) being powers of 2 important here).

Skipping details on computational aspects of \( \mathbb{F} \) which are closely tied to the proof of the theorem on fields.
### Chebyshev’s Inequality

For $a \geq 0$, $\Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$ equivalently for any $t > 0$, $\Pr[|X - E[X]| \geq t\sigma_X] \leq \frac{1}{t^2}$ where $\sigma_X = \sqrt{\text{Var}(X)}$ is the standard deviation of $X$.

Suppose $X = X_1 + X_2 + \ldots + X_n$. If $X_1, X_2, \ldots, X_n$ are independent then $\text{Var}(X) = \sum_i \text{Var}(X_i)$. 
Chebyshev’s Inequality

For $a \geq 0$, $\Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$ equivalently for any $t > 0$, $\Pr[|X - \mathbb{E}[X]| \geq t\sigma_X] \leq \frac{1}{t^2}$ where $\sigma_X = \sqrt{\text{Var}(X)}$ is the standard deviation of $X$.

Suppose $X = X_1 + X_2 + \ldots + X_n$. If $X_1, X_2, \ldots, X_n$ are independent then $\text{Var}(X) = \sum_i \text{Var}(X_i)$.

Lemma

Suppose $X = \sum_i X_i$ and $X_1, X_2, \ldots, X_n$ are pairwise independent, then $\text{Var}(X) = \sum_i \text{Var}(X_i)$.

Hence pairwise independence suffices if one relies only on Chebyshev inequality.
Part II

Hash Tables
Dictionary Data Structure

1. **U**: universe of keys with total order: numbers, strings, etc.

2. Data structure to store a subset $S \subseteq U$

3. **Operations:**
   - Search/look up: given $x \in U$ is $x \in S$?
   - Insert: given $x \not\in S$ add $x$ to $S$.
   - Delete: given $x \in S$ delete $x$ from $S$

4. **Static** structure: $S$ given in advance or changes very infrequently, main operations are lookups.

5. **Dynamic** structure: $S$ changes rapidly so inserts and deletes as important as lookups.

Can we do everything in $O(1)$ time?
Hashing and Hash Tables

Hash Table data structure:

1. A (hash) table/array $T$ of size $m$ (the table size).
2. A hash function $h : U \rightarrow \{0, \ldots, m - 1\}$.
3. Item $x \in U$ hashes to slot $h(x)$ in $T$. 

Given $S \subseteq U$. How do we store $S$ and how do we do lookups?

Ideal situation:

1. Each element $x \in S$ hashes to a distinct slot in $T$.
2. Lookup: Given $y \in U$ check if $T[h(y)] = y$. $O(1)$ time!

Collisions unavoidable if $|T| < |U|$.
Hashing and Hash Tables

Hash Table data structure:

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Given $S \subseteq U$. How do we store $S$ and how do we do lookups?

**Ideal situation:**

1. Each element $x \in S$ hashes to a distinct slot in $T$. Store $x$ in slot $h(x)$
2. **Lookup**: Given $y \in U$ check if $T[h(y)] = y$. $O(1)$ time!

Collisions unavoidable if $|T| < |U|$. 
Handling Collisions: Chaining

Collision: \( h(x) = h(y) \) for some \( x \neq y \).

Chaining/Open hashing to handle collisions:

1. For each slot \( i \) store all items hashed to slot \( i \) in a linked list. \( T[i] \) points to the linked list
2. Lookup: to find if \( y \in U \) is in \( T \), check the linked list at \( T[h(y)] \). Time proportion to size of linked list.

Does hashing give \( O(1) \) time per operation for dictionaries?
Hash Functions

Parameters: $N = |\mathcal{U}|$ (very large), $m = |T|$, $n = |S|$
Goal: $O(1)$-time lookup, insertion, deletion.

Single hash function

If $N \geq m^2$, then for any hash function $h : \mathcal{U} \rightarrow T$ there exists $i < m$ such that at least $N/m \geq m$ elements of $\mathcal{U}$ get hashed to slot $i$. Any $S$ containing all of these is a very very bad set for $h$! Such a bad set may lead to $O(m)$ lookup time!

In practice:

- Dictionary applications: choose a simple hash function and hope that worst-case bad sets do not arise
- Crypto applications: create “hard” and “complex” function very carefully which makes finding collisions difficult
Consider a family $\mathcal{H}$ of hash functions with good properties and choose $h$ randomly from $\mathcal{H}$.

- Guarantees: small $\#$ collisions in expectation for any given $S$.
- $\mathcal{H}$ should allow efficient sampling.
- Each $h \in \mathcal{H}$ should be efficient to evaluate and require small memory to store.

In other words a hash function is a “pseudorandom” function.
Strongly Universal Hashing

1. **Uniform**: Consider any element \( x \in \mathcal{U} \). Then if \( h \in \mathcal{H} \) is picked randomly then \( x \) should go into a random slot in \( T \). In other words \( \Pr[h(x) = i] = 1/m \) for every \( 0 \leq i < m \).

2. **(2)-Strongly Universal**: Consider any two distinct elements \( x, y \in \mathcal{U} \). Then if \( h \in \mathcal{H} \) is picked randomly then \( h(x) \) and \( h(y) \) should be independent random variables.
**Universal Hashing**

- **(2)-Universal:** Consider any two distinct elements $x, y \in U$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between $x$ and $y$ should be at most $1/m$. In other words $\Pr[h(x) = h(y)] \leq 1/m$.

Note: we do not insist on uniformity.
(2)-Universal: Consider any two distinct elements $x, y \in U$. Then if $h \in \mathcal{H}$ is picked randomly then the probability of a collision between $x$ and $y$ should be at most $1/m$. In other words $\Pr[h(x) = h(y)] \leq 1/m$.

Note: we do not insist on uniformity.

Universal hashing is a relaxation of strong universal hashing and simpler to construct while retaining most of the useful properties.
(Strongly) Universal Hashing

Definition

A family of hash functions $\mathcal{H}$ is (2-)strongly universal if for all distinct $x, y \in \mathcal{U}$, $h(x)$ and $h(y)$ are independent for $h$ chosen uniformly at random from $\mathcal{H}$, and for all $x$, $h(x)$ is uniformly distributed.

Definition

A family of hash functions $\mathcal{H}$ is (2-)universal if for all distinct $x, y \in \mathcal{U}$, $Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq 1/m$ where $m$ is the table size.
Analyzing Universal Hashing

1. $T$ is hash table of size $m$.
2. $S \subseteq U$ is a fixed set of size $n$.
3. $h$ is chosen randomly from a universal hash family $\mathcal{H}$.
4. $x$ is a fixed element of $U$.

**Question:** What is the expected time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?
Analyzing Universal Hashing

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**Question:** What is the expected time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?

1. The time to look up $x$ is the size of the list at $T[h(x)]$: same as the number of elements in $S$ that collide with $x$ under $h$.
2. $\ell(x)$ be this number. We want $E[\ell(x)]$
3. Let $C_{x,y}$ be indicator random variable for $x, y$ colliding under $h$, that $C_{x,y} = 1$ iff $h(x) = h(y)$
Analyzing Universal Hashing

Continued...

Number of elements colliding with $x$: $\ell(x) = \sum_{y \in S} C_{x,y}$.

$$\Rightarrow E[\ell(x)] = \sum_{y \in S, y \neq x} E[C_{x,y}]$$

linearity of expectation

$$= \sum_{y \in S, y \neq x} Pr[h(x) = h(y)]$$

$$\leq \sum_{y \in S, y \neq x} \frac{1}{m}$$

(since $\mathcal{H}$ is a universal hash family)

$$\leq \frac{|S|}{m}$$

$$\leq \frac{n}{m}$$

$$\leq 1$$

(if $|S| \leq m$)
Analyzing Universal Hashing

Comments:

1. Expected time for insertion and deletion also $O(1)$ if $n \leq m$.

2. Analysis assumes static set $S$ but holds as long as $S$ is a set formed with at most $O(m)$ insertions and deletions. Assumption is that insertions and deletions are not adaptive.

3. **Worst-case**: look up time can be large! How large? Technically $O(n)$ if all elements collide.
If \( h \) is a fully random function and \( m = n \) then expected maximum load in any bucket of \( T \) is \( O(\log n / \log \log n) \) via balls and bin analogy.

If \( h \) is chosen from a universal hash family \( \mathcal{H} \) what is the expected maximum load?

**Lemma**

Let \( h \) be chosen from a universal hash family and let \( m \geq n \) and let \( L \) be maximum load of any slot. Then \( \Pr[L > t\sqrt{n}] \leq 1/t^2 \) for \( t \geq 1 \).
If $h$ is a fully random function and $m = n$, then expected maximum load in any bucket of $T$ is $O(\log n / \log \log n)$ via balls and bin analogy.

If $h$ is chosen from a universal hash family $\mathcal{H}$, what is the expected maximum load?

**Lemma**

Let $h$ be chosen from a universal hash family and let $m \geq n$ and let $L$ be maximum load of any slot. Then $\Pr[L > t\sqrt{n}] \leq 1/t^2$ for $t \geq 1$.

Thus $L = O(\sqrt{n})$ with probability at least $1/2$. 
Lemma

Let \( h \) be chosen from a universal hash family and let \( m \geq n \) and let \( L \) be maximum load of any slot. Then \( \Pr[L > t\sqrt{n}] \leq 1/t^2 \) for \( t \geq 1 \).

Let \( C = \sum_{x,y \in S, x \neq y} C_{x,y} \) be total number of collisions.

- \( \mathbb{E}[C] \leq \binom{n}{2}/m \leq (n - 1)/2 \) if \( m \geq n \).

Observation: \( C \geq \binom{L}{2} \). Why?

- \( L > t\sqrt{n} \) implies \( C > t^2 n/2 \).
- By Markov \( \Pr[C > t^2 n/2] \leq \mathbb{E}[C] / (t^2 n/2) \leq 1/t^2 \)
- Hence \( \Pr[L > t\sqrt{n}] \leq 1/t^2 \).
Analyzing Universal Hashing: Maximum Load

Lemma

Let $h$ be chosen from a universal hash family and let $m \geq n$ and let $L$ be maximum load of any slot. Then $E[L] = O(\sqrt{n})$.

Direct proof: $(E[L])^2 \leq E[L^2] \leq E[C] \leq n$ (using Jensen’s ineq)

$L$ is a non-negative random variable in range. Hence

$$E[L] = \sum_{i=1}^{n} \Pr[L \geq i] \quad \text{(from defn of expectation)}$$

$$\leq \sqrt{n} + \sum_{i=1}^{\sqrt{n}} 1 + \sum_{i=\sqrt{n}+1}^{n} n/i^2 \quad \text{(from previous lemma)}$$

$$\leq \sqrt{n} + n \int_{\sqrt{n}}^{n} 1/i^2 \leq 2\sqrt{n}.$$
Compact Strongly Universal Hash Family

Parameters: \( N = |\mathcal{U}|, \ m = |T|, \ n = |S| \)

**Question:** How do we construct strongly universal hash family?
Compact Strongly Universal Hash Family

Parameters: $N = |U|$, $m = |T|$, $n = |S|$

**Question:** How do we construct strongly universal hash family?

If $N$ and $m$ are powers of 2 then use construction of $N$ pairwise independent random variables over range $[m]$ discussed previously.
Compact Strongly Universal Hash Family

Parameters: \( N = |\mathcal{U}|, \ m = |T|, \ n = |S| \)

**Question:** How do we construct strongly universal hash family?

If \( N \) and \( m \) are powers of 2 then use construction of \( N \) pairwise independent random variables over range \([m]\) discussed previously.

**Disadvantage:** Need \( m \) to be power of 2 and requires complicated field operations.
Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |\mathcal{T}|$, $n = |\mathcal{S}|$

1. Choose a prime number $p > N$. Define function $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.

2. Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$ ($\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$). Note that $|\mathcal{H}| = p(p - 1)$.
Compact Universal Hash Family

Parameters: \( N = |U|, \ m = |T|, \ n = |S| \)

1. Choose a prime number \( p > N \). Define function
   \( h_{a,b}(x) = ((ax + b) \mod p) \mod m. \)
2. Let \( \mathcal{H} = \{ h_{a,b} | \ a, b \in \mathbb{Z}_p, a \neq 0 \} \) \( (\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}) \). Note that \( |\mathcal{H}| = p(p - 1). \)

Theorem

\( \mathcal{H} \ is \ a \ universal \ hash \ family. \)
Compact Universal Hash Family

Parameters: \( N = |U|, m = |T|, n = |S| \)

1. Choose a prime number \( p > N \). Define function \( h_{a,b}(x) = ((ax + b) \mod p) \mod m \).

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Theorem

\( \mathcal{H} \) is a universal hash family.

Comments:

1. \( h_{a,b} \) can be evaluated in \( O(1) \) time.

2. Easy to store, i.e., just store \( a, b \). Easy to sample.
Understanding the hashing

Once we fix $a$ and $b$, and we are given a value $x$, we compute the hash value of $x$ in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

Let $g_{a,b}(x) = (ax + b) \mod p$.

$h_{a,b}(x) = g_{a,b}(x) \mod m$. 
Understanding the hashing

Once we fix $a$ and $b$, and we are given a value $x$, we compute the hash value of $x$ in two stages:

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$h_{a,b}(x) = g_{a,b}(x) \mod m$.

Fix $x$:

- $g_{a,b}(x)$ is uniformly distributed in $\{0, 1, \ldots, p - 1\}$. Why?
- However $h_{a,b}(x)$ is not necessarily uniformly distributed over $\{0, 1, 2, \ldots, m\}$. Why?
Some math required...

Recall $\mathbb{Z}_p$ is a field.

- $a \neq 0$ implies unique $a'$ such that $aa' = 1 \mod p$
- For $a, x, y \in \mathbb{Z}_p$ such that $x \neq y$ and $a \neq 0$ we have $ax \neq ay \mod p$.
- For $x \neq y$ and any $r, s$ there is a unique solution $(a, b)$ to the equations $ax + b = r$ and $ay + b = s$. 
Proof of the Theorem: Outline

\[ h_{a,b}(x) = ((ax + b) \mod p) \mod m. \]

**Theorem**

\[ \mathcal{H} = \{ h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0 \} \text{ is universal.} \]

**Proof.**

Fix \( x, y \in \mathcal{U}, x \neq y \). Show that

\[ \Pr_{h_{a,b} \sim \mathcal{H}}[h_{a,b}(x) = h_{a,b}(y)] \leq 1/m. \]

Note that \( |\mathcal{H}| = p(p - 1). \)
Proof of the Theorem: Outline

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Note that \( |\mathcal{H}| = p(p-1) \).

1. Let \((a, b)\) (equivalently \(h_{a,b}\)) be *bad* for \(x, y\) if \( h_{a,b}(x) = h_{a,b}(y) \).
Proof of the Theorem: Outline

\[ h_{a,b}(x) = ((a x + b) \mod p) \mod m. \]

**Theorem**

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Note that \( |\mathcal{H}| = p(p-1) \).

1. Let \((a, b)\) (equivalently \(h_{a,b}\)) be bad for \(x, y\) if \(h_{a,b}(x) = h_{a,b}(y)\).

2. **Claim:** Number of bad \((a, b)\) is at most \(p(p - 1)/m\).
Proof of the Theorem: Outline

\[ h_{a,b}(x) = ((ax + b) \mod p) \mod m. \]

**Theorem**
\[ \mathcal{H} = \{ h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0 \} \text{ is universal.} \]

**Proof.**
Fix \( x, y \in \mathcal{U}, x \neq y. \) Show that\[
\Pr_{h_{a,b} \sim \mathcal{H}}[h_{a,b}(x) = h_{a,b}(y)] \leq 1/m.
\]
Note that \( |\mathcal{H}| = p(p-1). \)

1. Let \((a, b)\) (equivalently \(h_{a,b}\)) be bad for \(x, y\) if \(h_{a,b}(x) = h_{a,b}(y)\).
2. **Claim:** Number of bad \((a, b)\) is at most \(p(p - 1)/m\).
3. Total number of hash functions is \(p(p - 1)\) and hence probability of a collision is \(\leq 1/m\).
Proof of Claim

\[ h_{a,b}(x) = (((ax + b) \mod p) \mod m) \]

2 lemmas ...

Fix \( x \neq y \in \mathbb{Z}_p \), and let \( r = (ax + b) \mod p \) and \( s = (ay + b) \mod p \).
Proof of Claim

\[ h_{a,b}(x) = (((ax + b) \mod p) \mod m) \]

2 lemmas ...

Fix \( x \neq y \in \mathbb{Z}_p \), and let \( r = (ax + b) \mod p \) and \( s = (ay + b) \mod p \).

1. 1-to-1 correspondence between \( p(p - 1) \) pairs of \((a, b)\) (equivalently \( h_{a,b} \)) and \( p(p - 1) \) pairs of \((r, s)\).
Proof of Claim

\[ h_{a,b}(x) = (((ax + b) \mod p) \mod m) \]

2 lemmas ...

Fix \( x \neq y \in \mathbb{Z}_p \), and let \( r = (ax + b) \mod p \) and \( s = (ay + b) \mod p \).

1. 1-to-1 correspondence between \( p(p - 1) \) pairs of \((a, b)\) (equivalently \( h_{a,b} \)) and \( p(p - 1) \) pairs of \((r, s)\).

2. Out of all possible \( p(p - 1) \) pairs of \((r, s)\), at most \( p(p - 1)/m \) fraction satisfies \( r \mod m = s \mod m \).
Correspondence Lemma

Lemma

If \( x \neq y \) then for each \((r, s)\) such that \( r \neq s \) and \( 0 \leq r, s \leq p-1 \) there is exactly one pair \((a, b)\) such that \( a \neq 0 \) and

\[
ax + b \mod p = r \quad \text{and} \quad ay + b \mod p = s.
\]

Proof.

Solve the two equations:

\[
ax + b = r \mod p \quad \text{and} \quad ay + b = s \mod p
\]

We get \( a = \frac{r-s}{x-y} \mod p \) and \( b = r - ax \mod p \).

One-to-one correspondence between \((a, b)\) and \((r, s)\).
Collisions due to folding

Once we fix $a$ and $b$, and we are given a value $x$, we compute the hash value of $x$ in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

Collision...

Given two distinct values $x$ and $y$ they might collide only because of folding.
Collisions due to folding

Once we fix $a$ and $b$, and we are given a value $x$, we compute the hash value of $x$ in two stages:

1. **Compute:** $r \leftarrow (ax + b) \mod p$.
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Collision...

Given two distinct values $x$ and $y$ they might collide only because of folding.

Lemma

The number of pairs $(r, s)$ of $\mathbb{Z}_p \times \mathbb{Z}_p$ such that $r \neq s$ and $r \mod m = s \mod m$ is at most $p(p - 1)/m$. 
Folding numbers

Lemma

\# pairs \((r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p\) such that \(r \neq s\) and \(r \mod m = s \mod m\) (folded to the same number) is \(p(p - 1)/m\).

Proof.

Consider a pair \((r, s) \in \{0, 1, \ldots, p - 1\}^2\) s.t. \(r \neq s\). Fix \(r\):

1. Let \(d = r \mod m\).
Folding numbers

Lemma

\( \# \text{ pairs } (r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p \text{ such that } r \neq s \text{ and } r \mod m = s \mod m \text{ (folded to the same number) is } p(p - 1)/m. \)

Proof.

Consider a pair \((r, s) \in \{0, 1, \ldots, p - 1\}^2\) s.t. \(r \neq s\). Fix \(r\):

1. Let \(d = r \mod m\).
2. There are \(\lceil p/m \rceil\) values of \(s\) such that \(r \mod m = s \mod m\).
3. One of them is when \(r = s\).
4. \(\implies\) \# of colliding pairs
Folding numbers

Lemma

# pairs \((r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p\) such that \(r \neq s\) and \(r \mod m = s \mod m\) (folded to the same number) is \(p(p - 1)/m\).

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Consider a pair \((r, s) \in \{0, 1, \ldots, p - 1\}^2\) s.t. \(r \neq s\). Fix \(r\):

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2. There are \(\lceil p/m \rceil\) values of \(s\) such that \(r \mod m = s \mod m\).
3. One of them is when \(r = s\).
4. \(\implies\) # of colliding pairs \((\lceil p/m \rceil - 1)p \leq (p - 1)p/m\)
Proof of Claim
# of bad pairs is \( p(p - 1)/m \)

Proof.

Let \( a, b \in \mathbb{Z}_p \) such that \( a \neq 0 \) and \( h_{a,b}(x) = h_{a,b}(y) \).

1. Let \( r = ax + b \mod p \) and \( s = ay + b \mod p \).

2. Collision if and only if \( r \mod m = s \mod m \).

3. (Folding error): Number of pairs \( (r, s) \) such that \( r \neq s \) and \( 0 \leq r, s \leq p - 1 \) and \( r \mod m = s \mod m \) is \( p(p - 1)/m \).

4. From previous lemma there is one-to-one correspondence between \( (a, b) \) and \( (r, s) \). Hence total number of bad \( (a, b) \) pairs is \( p(p - 1)/m \).
Proof of Claim

# of bad pairs is $p(p - 1)/m$

**Proof.**

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

1. Let $r = ax + b \mod p$ and $s = ay + b \mod p$.

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3. (Folding error): Number of pairs $(r, s)$ such that $r \neq s$ and $0 \leq r, s \leq p - 1$ and $r \mod m = s \mod m$ is $p(p - 1)/m$.

4. From previous lemma there is one-to-one correspondence between $(a, b)$ and $(r, s)$. Hence total number of bad $(a, b)$ pairs is $p(p - 1)/m$.

**Prob of $x$ and $y$ to collide:** \[
\frac{\# \text{ bad } (a, b) \text{ pairs}}{\#(a, b) \text{ pairs}} = \frac{p(p-1)/m}{p(p-1)} = \frac{1}{m}.
\]
Part III

Perfect Hashing
**Perfect Hashing**

**Question:** Suppose we get a set $S \subset \mathcal{U}$ of size $n$. Can we design an “efficient” and “perfect” hash function?

- Create a table $T$ of size $m = O(n)$.
- Create a hash function $h : S \rightarrow [m]$ with no collisions!
- $h$ should be fast and efficient to evaluate
- Construct $h$ efficiently given $S$. Construction of $h$ can be randomized (Las Vegas algorithm)

A perfect hash function would guarantee lookup time of $O(1)$.
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Suppose $m = n^2$. Table size is much bigger than $n$

**Lemma**

Suppose $\mathcal{H}$ is a universal hash family and $m = n^2$. Then

$$\Pr_{h \in \mathcal{H}}[\text{no collisions in } S] \geq 1/2.$$
Perfect Hashing via Large Space

Suppose $m = n^2$. Table size is much bigger than $n$

Lemma

Suppose $\mathcal{H}$ is a universal hash family and $m = n^2$. Then $\Pr_{h \in \mathcal{H}}[\text{no collisions in } S] \geq 1/2$.

Proof.

- Total number of collisions is $C = \sum_{x, y \in S, x \neq y} C_{x, y}$.
- $\mathbb{E}[C] \leq \binom{n}{2}/m < 1/2$.
- By Markov inequality $\Pr[C \geq 1] < 1/2$. 
Perfect Hashing via Large Space

Suppose $m = n^2$. Table size is much bigger than $n$

Lemma

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- By Markov inequality $\Pr[C \geq 1] < 1/2$.

Algorithm: pick $h \in \mathcal{H}$ randomly and check if $h$ is perfect. Repeat until success.
Perfect Hashing

Two levels of hash tables

**Question:** Can we obtain perfect hashing with \( m = O(n) \)?

**Perfect Hashing**

- Do hashing once with table \( T \) of size \( m \).

- For each slot \( i \) in \( T \) let \( Y_i \) be number of elements hashed to slot \( i \). If \( Y_i > 1 \) use perfect hashing with second table \( T_i \) of size \( Y_i^2 \).
**Question:** Can we obtain perfect hashing with $m = O(n)$?

**Perfect Hashing**

- Do hashing once with table $T$ of size $m$
- For each slot $i$ in $T$ let $Y_i$ be number of elements hashed to slot $i$. If $Y_i > 1$ use perfect hashing with second table $T_i$ of size $Y_i^2$.

Construction gives perfect hashing. What is the space used?
Perfect Hashing
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Construction gives perfect hashing. What is the space used?

$$Z = m + \sum_{i=0}^{m-1} Y_i^2$$

a random variable (depends on random choice of first level hash function)
Perfect Hashing

\( O(n) \) space usage

\( h \) the primary random hash function.
Perfect Hashing

$O(n)$ space usage

$h$ the primary random hash function.

Claim

$$E\left[\sum_{i=0}^{m-1} Y_i^2\right] \leq \frac{3n}{2} \text{ if } m \geq n.$$
Perfect Hashing

O(n) space usage

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E \left[ \sum_{i=0}^{m-1} Y_i^2 \right] \leq 3n/2 \text{ if } m \geq n.
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**Proof.**

Let \( C \) be total number of collisions. We already saw \( E[C] \leq \binom{n}{2}/m \).
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O(n) space usage

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**Claim**

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\]

**Proof.**

Let \( C \) be the total number of collisions. We already saw \( E[C] \leq \binom{n}{2}/m \).

\[
\sum_i \binom{Y_i}{2} = C \quad \text{and hence} \quad \sum_i Y_i^2 = 2C + \sum_i Y_i.
\]

Therefore

\[
E \left[ \sum_i Y_i^2 \right] \leq 2 \binom{n}{2}/m + E \left[ \sum_i Y_i \right] = 2 \binom{n}{2}/m + n \leq 3n/2.
\]
Perfect Hashing

Do hashing once with table $T$ of size $m$

For each slot $i$ in $T$ let $Y_i$ be number of elements hashed to slot $i$. If $Y_i > 1$ use perfect hashing with second table $T_i$ of size $Y_i^2$.

Space usage is $Z = m + \sum_{i=0}^{m-1} Y_i^2$ and $\mathbb{E}[Z] \leq 5n/2$ if $m = n$.

Use algorithm to create perfect hash table

By Markov space usage is $< 5n$ with probability at least $1/2$

Repeat if space usage is larger than $5n$. Expected number of repetitions is 2. Hence it leads to $O(n)$ time Las Vegas algorithm

Technically also need to count the space to store multiple hash functions: $O(n)$ overhead
Rehashing, amortization and...
... making the hash table dynamic

So far we assumed fixed $S$ of size $\sim m$.

**Question:** What happens as items are inserted and deleted?

1. If $|S|$ grows to more than $cm$ for some constant $c$ then hash table performance clearly degrades.

2. If $|S|$ stays around $\sim m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

**Solution:** Rebuild hash table periodically!

1. Choose a new table size based on current number of elements in the table.
2. Choose a new random hash function and rehash the elements.
3. Discard old table and hash function.

**Question:** When to rebuild? How expensive?
Rehashing, amortization and...  
... making the hash table dynamic

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**Question:** When to rebuild? How expensive?
Rebuilding the hash table

1. Start with table size $m$ where $m$ is some estimate of $|S|$ (can be some large constant).

2. If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.
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3. If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant $c$ (say 10), rebuild.
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The amortize cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when $S$ changes. Hence $O(1)$ expected look up/insert/delete time dynamic data dictionary data structure!
Practical Issues

Hashing used typically for integers, vectors, strings etc.

- Universal hashing is defined for integers. To implement for other objects need to map objects in some fashion to integers (via representation)

- Practical methods for various important cases such as vectors, strings are studied extensively. See http://en.wikipedia.org/wiki/Universal_hashing for some pointers.


Part IV

Bloom Filters
Bloom Filters

Hashing:
1. To insert $x$ in dictionary store $x$ in table in location $h(x)$
2. To lookup $y$ in dictionary check contents of location $h(y)$
3. Storing items in dictionary expensive in terms of memory, especially if items are unwieldy objects such a long strings, images, etc with non-uniform sizes.
Bloom Filters

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Bloom Filter: tradeoff space for false positives

1. To insert $x$ in dictionary set bit to 1 in location $h(x)$ (initially all bits are set to 0)
2. To lookup $y$ if bit in location $h(y)$ is 1 say yes, else no.
Bloom Filters

**Bloom Filter**: tradeoff space for false positives

1. To insert \( x \) in dictionary set *bit* to 1 in location \( h(x) \) (initially all bits are set to 0)
2. To lookup \( y \) if bit in location \( h(y) \) is 1 say yes, else no
3. No false negatives but false positives possible due to collisions
Bloom Filters

**Bloom Filter:** tradeoff space for false positives

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Reducing false positives:

1. Pick $k$ hash functions $h_1, h_2, \ldots, h_k$ independently
Bloom Filters

**Bloom Filter:** tradeoff space for false positives

1. To insert \( x \) in dictionary set *bit* to \( 1 \) in location \( h(x) \) (initially all bits are set to \( 0 \))
2. To lookup \( y \) if bit in location \( h(y) \) is \( 1 \) say yes, else no
3. No false negatives but false positives possible due to collisions

Reducing false positives:

1. Pick \( k \) hash functions \( h_1, h_2, \ldots, h_k \) *independently*
2. To insert set \( h_i(x) \)th bit to one in table \( i \) for each \( 1 \leq i \leq k \)
Bloom Filters

Bloom Filter: tradeoff space for false positives

1. To insert $x$ in dictionary set bit to 1 in location $h(x)$ (initially all bits are set to 0)
2. To lookup $y$ if bit in location $h(y)$ is 1 say yes, else no
3. No false negatives but false positives possible due to collisions

Reducing false positives:

1. Pick $k$ hash functions $h_1, h_2, \ldots, h_k$ independently
2. To insert set $h_i(x)$th bit to one in table $i$ for each $1 \leq i \leq k$
3. To lookup $y$ compute $h_i(y)$ for $1 \leq i \leq k$ and say yes only if each bit in the corresponding location is 1, otherwise say no. If probability of false positive for one hash function is $\alpha < 1$ then with $k$ independent hash function it is $\alpha^k$. 