CS 473: Algorithms, Fall 2018

Fast Fourier Transform

Lecture 5

September 13, 2018

5.1: Introduction

What is going on?

Clicker question

Consider the formula $\sqrt{xy} = \sqrt{x}\sqrt{y}$.
$\implies 1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = -1$.

1. $1 = -1$. Its time that this was more publicly known.
2. The formula $\sqrt{xy} = \sqrt{x}\sqrt{y}$ is incorrect.
3. $\sqrt{-1}$ is two numbers, and the above formula is incorrect in this case.
4. Wikipedia knows the answer.
5. This is not related to the class topic, so stop wasting my time.

Polynomials of degree $\mathbf{2}$

Clicker question

Consider the polynomial $p(x) = ax^2 + bx + c$ that passes through the points $(0, 1), \quad (-1, 1), \quad (1, 2)$. Which of the following statements are correct?

1. There are infinite family of such polynomials.
2. There is no such polynomial.
3. There is only one such polynomial, but its coefficients are complex numbers.
4. There is only one such polynomial, and it is $p(x) = x^2/2 + x/2 + 1$.
5. None of the above.

Polynomials of degree n

Clicker question

Consider two polynomials $p(x) = \sum_{i=0}^{n-1} a_i x^i$ and $q(x) = \sum_{i=0}^{n-1} b_i x^i$ that passes through the points (x_i, y_i), for $i = 1, \ldots, n$. Then:

1. $p(x) = q(x)$, for all x.
2. $p(x) \neq q(x)$, for all $x \in \mathbb{R} \setminus \{x_1, \ldots, x_n\}$.
3. Both (A) and (B) are possible.
4. None of the above.

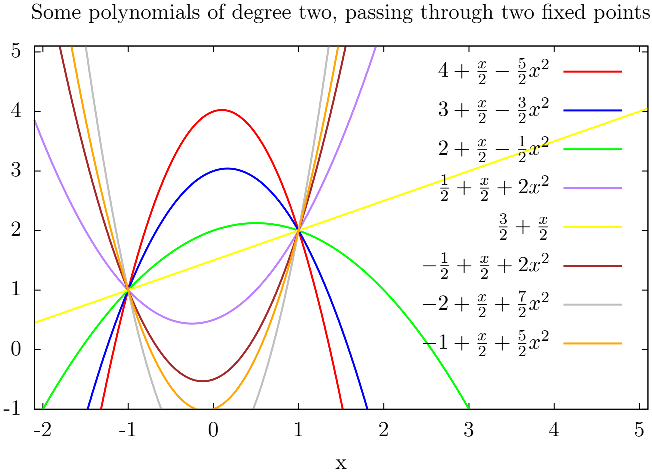
Approximating functions with polynomials

Clicker question

Let f be a continuous function on the interval $[0, 1]$.
Let $\varepsilon > 0$ be a parameter. Then, we have:

1. $\exists n > 0$, and a polynomial $p(x)$ of degree n, such that $\forall x \in [0, 1] \quad |p(x) - f(x)| \leq \varepsilon$.
2. For $n = O(1/\varepsilon^2)$, there exists a polynomial $p(x)$ of degree n, such that $\forall x \in [0, 1] \quad |p(x) - f(x) \leq \varepsilon$.
3. There might not be a polynomial that can approximate f on $[0, 1]$, up to additive error of ε.
4. None of the above.

Polynomials and point value pairs



Multiplying polynomials quickly

Definition

polynomial $p(x)$ of degree n:a function
$p(x) = \sum_{j=0}^{n} a_j x^j = a_0 + x(a_1 + x(a_2 + \ldots + x a_n))$.
x_0: $p(x_0)$ can be computed in $O(n)$ time.
"dual" (and equivalent) representation...

Theorem

For any set $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$ *of* n ***point-value pairs*** *such that all the* x_k *values are distinct, there is a unique polynomial* $p(x)$ *of degree* $n - 1$, *such that* $y_k = p(x_k)$, *for* $k = 0, \ldots, n - 1$.

Polynomial via point-value

Clicker question

Let $x_0, \ldots, x_n$ be $n+1$ distinct real numbers.

$$p(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

1. $p(x)$ is a polynomial of degree n, we have $p(x_0) = 0$, and $p(x_1) = 1, p(x_2) = 1, \ldots, p(x_n) = 1$.
2. $p(x)$ is a rational function.
3. $p(x)$ is a polynomial of degree n, we have $p(x_0) = 1$, and $p(x_1) = 0, p(x_2) = 0, \ldots, p(x_n) = 0$.
4. $p(x)$ is not well defined function because of division by zero.

Polynomial via point-value

$\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}$: polynomial through points:

$$p(x) = y_0 \frac{\cancel{(x - x_0)}(x - x_1)(x - x_2)}{\cancel{(x_0 - x_0)}(x_0 - x_1)(x_0 - x_2)}$$

$$+ y_1 \frac{(x - x_0)\cancel{(x - x_1)}(x - x_2)}{(x_1 - x_0)\cancel{(x_1 - x_1)}(x_1 - x_2)}$$

$$+ y_2 \frac{(x - x_0)(x - x_1)\cancel{(x - x_2)}}{(x_2 - x_0)(x_2 - x_1)\cancel{(x_2 - x_2)}}$$

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Polynomial via point-value

$\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$: polynomial through points:

$$p(x) = \sum_{i=0}^{n-1} y_i \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$

ith is zero for $x = x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}$, and is equal to y_i for $x = x_i$.

Polynomials: regular vs. point-value pair representation

Just because.

1. Given n point-value pairs. Can compute $p(x)$ in $O(n^2)$ time.
2. Point-value pairs representation: Multiply polynomials quickly!
3. p, q polynomial of degree $n-1$, both represented by $2n$ point-value pairs

$$\left\{(x_0, y_0), (x_1, y_1), \ldots, (x_{2n-1}, y_{2n-1})\right\} \quad \text{for} \quad p(x),$$
$$\text{and} \quad \left\{(x_0, y_0'), (x_1, y_1'), \ldots, (x_{2n-1}, y_{2n-1}')\right\} \quad \text{for} \quad q(x).$$

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Polynomials: regular vs. point-value pair representation

Just because.

1. In point-value representation representation of $r(x)$ is

$$\begin{aligned}
&\left\{(x_0, r(x_0)), \ldots, (x_{2n-1}, r(x_{2n-1}))\right\} \\
&\qquad = \left\{\left(x_0, p(x_0)q(x_0)\right), \ldots, \left(x_{2n-1}, p(x_{2n-1})q(x_{2n-1})\right)\right\} \\
&\qquad = \left\{(x_0, y_0 y'_0), \ldots, (x_{2n-1}, y_{2n-1} y'_{2n-1})\right\}.
\end{aligned}$$

Which implies...

1. $p(x)$ and $q(x)$: point-value pairs $\implies$ compute $r(x) = p(x)q(x)$ in linear time!
2. ...but $r(x)$ is in point-value representation. Bummer.
3. ...but we can compute $r(x)$ from this representation.
4. Purpose: Translate quickly (i.e., $O(n \log n)$ time) from the standard r to point-value pairs representation of polynomials.
5. ...and back!
6. $\implies$ computing product of two polynomials in $O(n \log n)$ time.
7. ***Fast Fourier Transform*** is a way to do this.

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5.2: Computing a polynomial quickly on n values

Computing a polynomial quickly on n values

Lets just use some magic.

1. Assume: polynomials have degree $n-1$, where $n = 2^k$.
2. .. pad polynomials with terms having zero coefficients.
3. ***Magic set*** of numbers: $\Psi = \{x_1, \ldots, x_n\}$.
   Property: $|\mathbf{SQ}(\Psi)| = n/2$, where $\mathbf{SQ}(\Psi) = \left\{ x^2 \;\middle|\; x \in \Psi \right\}$.
4. $|\mathrm{square}()| = |\Psi|/2$.
5. Easy to find such set...
6. Magic: Have this property repeatedly... $\mathbf{SQ}(\mathbf{SQ}(\Psi))$ has $n/4$ distinct values.
7. $\mathbf{SQ}(\mathbf{SQ}(\mathbf{SQ}(\Psi)))$ has $n/8$ values.
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Collapsible sets

Assume magic...

Let us for the time being ignore this technicality, and fly, for a moment, into the land of fantasy, and assume that we do have such a set of numbers, so that $|\mathsf{SQ}^i(\Psi)| = n/2^i$ numbers, for $i = 0, \ldots, k$. Let us call such a set of numbers ***collapsible***.

Breaking the input polynomial into...

... two polynomials of half the degree

1. For a set $\mathcal{X} = \{x_0, \ldots, x_n\}$ and polynomial $p(x)$, let

$$p\Big(\mathcal{X}\Big) = \Big\langle (x_0, p(x_0)), \ldots, (x_n, p(x_n)) \Big\rangle .$$

2. $p(x) = \sum_{i=0}^{n-1} a_i x^i$ as $p(x) = u(x^2) + x \cdot v(x^2)$, where

$$u(y) = \sum_{i=0}^{n/2-1} a_{2i} y^i \qquad \text{and} \qquad v(y) = \sum_{i=0}^{n/2-1} a_{1+2i} y^i.$$

3. all even degree terms in $u(\cdot)$, all odd degree terms in $v(\cdot)$

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FFT: The dividing stage

1. $p(x) = \sum_{i=0}^{n-1} a_i x^i$ as
   $p(x) = u(x^2) + x \cdot v(x^2)$.
2. Ψ: collapsible set of size n.
3. $p(\Psi)$: compute polynomial of degree $n-1$ on n values.
4. Decompose:

$$u(y) = \sum_{i=0}^{n/2-1} a_{2i} y^i \quad \text{and} \quad v(y) = \sum_{i=0}^{n/2-1} a_{1+2i} y^i.$$

5. Need to compute $u(x^2)$, for all $x \in \Psi$.
6. Need to compute $v(x^2)$, for all $x \in \Psi$.
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FFT algorithm

```
FFTAlg(p, X)// X: A collapsible set of n elements.
    input: p(x): polynomial deg. n: p(x) = sum_{i=0}^{n-1} a_i x^i
    output: p(X)
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$u(y) = \sum_{i=0}^{n/2-1} a_{2i} y^i \qquad v(y) = \sum_{i=0}^{n/2-1} a_{1+2i} y^i .$

$Y = \mathbf{SQ}(X) = \left\{ x^2 \;\middle|\; x \in X \right\}.$

$U = \mathbf{FFTAlg}(u,\ Y)$ // $U = u(Y)$

$V = \mathbf{FFTAlg}(v,\ Y)$ // $V = v(Y)$

$Out \leftarrow \emptyset$

for $x \in X$ **do** // $p(x) = u(x^2) + x * v(x^2)$

 $(x, p(x)) \leftarrow (x,\ U[x^2] + x \cdot V[x^2])$ // $U[x^2] \equiv u(x^2)$

 $Out \leftarrow Out \cup \{(x, p(x))\}$

return Out

Running time analysis...

...an old foe emerges once again to serve

1. $T(m, n)$: Time of computing a polynomial of degree m on n values.
2. We have that:

$$T(n-1, n) = 2T(n/2-1, n/2) + O(n).$$

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2. Trick: Use complex numbers!

Complex numbers – a quick reminder

1. Complex number: pair (α, β) of real numbers. Written as $\tau = \alpha + \mathbf{i}\beta$.
2. α: ***real*** part, β: ***imaginary*** part.
3. $\mathbf{i}$ is the root of -1.
4. Geometrically: a point in the complex plane:

1. ***polar form***: $\tau = r\cos\phi + \mathbf{i}r\sin\phi = r(\cos\phi + \mathbf{i}\sin\phi)$

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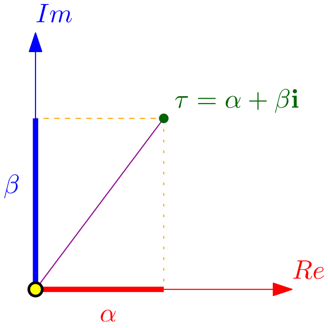

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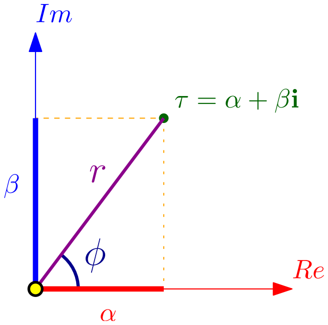

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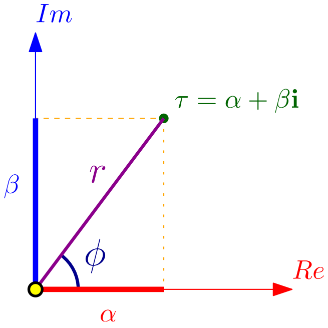

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A useful formula: $\cos\phi + \mathrm{i}\sin\phi = \mathrm{e}^{\mathrm{i}\phi}$

1. By Taylor's expansion:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\,,$$

and $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\,.$$

2. Since $\mathrm{i}^2 = -1$:

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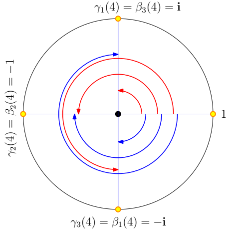
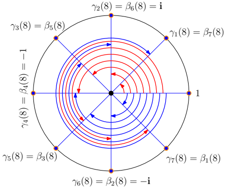
Roots of unity

The desire to avoid war?

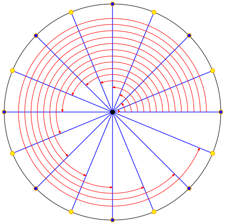
For $j = 0, \ldots, n-1$, we get the n distinct ***roots of unity***.



$(n = 4)$



$(n = 8)$



$(n = 16)$

Back to collapsible sets

1. Can do all basic calculations on complex numbers in $O(1)$ time.
2. Idea: Work over the complex numbers.
3. Use roots of unity!
4. γ: nth root of unity. There are n such roots, and let $\gamma_j(n)$ denote the jth root.

   $$\gamma_j(n) = \cos((2\pi j)/n) + \mathrm{i}\sin((2\pi j)/n) = \gamma^j.$$

   Let $\mathcal{A}(n) = \{\gamma_0(n), \ldots, \gamma_{n-1}(n)\}$.
5. $|\mathbf{SQ}(\mathcal{A}(n))|$ has $n/2$ entries.
6. $\mathbf{SQ}(\mathcal{A}(n)) = \mathcal{A}(n/2)$
7. n to be a power of 2, then $\mathcal{A}(n)$ is the *required*

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The first result...

Theorem

Given polynomial $p(x)$ of degree n, where n is a power of two, then we can compute $p(X)$ in $O(n \log n)$ time, where $X = \mathcal{A}(n)$ is the set of n different powers of the nth root of unity over the complex numbers.

Problem...

We can go, but can we come back?

1. Can multiply two polynomials quickly
2. by transforming them to the point-value pairs representation...
3. over the nth roots of unity.
4. Q: How to transform this representation back to the regular representation.
5. A: Do some confusing math...

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5.3: Recovering the polynomial

Recovering the polynomial

Think about FFT as a matrix multiplication operator.
$p(x) = \sum_{i=0}^{n-1} a_i x^i$. Evaluating $p(\cdot)$ on $\mathcal{A}(n)$:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \gamma_0 & \gamma_0^2 & \gamma_0^3 & \cdots & \gamma_0^{n-1} \\ 1 & \gamma_1 & \gamma_1^2 & \gamma_1^3 & \cdots & \gamma_1^{n-1} \\ 1 & \gamma_2 & \gamma_2^2 & \gamma_2^3 & \cdots & \gamma_2^{n-1} \\ 1 & \gamma_3 & \gamma_3^2 & \gamma_3^3 & \cdots & \gamma_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \gamma_{n-1} & \gamma_{n-1}^2 & \gamma_{n-1}^3 & \cdots & \gamma_{n-1}^{n-1} \end{pmatrix}}_{\text{the matrix } V} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix},$$

where $\gamma_j = \gamma_j(n) = (\gamma_1(n))^j$ is the jth power of the

The Vandermonde matrix

Because every matrix needs a name

$\boldsymbol{V}$ is the ***Vandermonde*** matrix.
$\boldsymbol{V^{-1}}$: inverse matrix of $\boldsymbol{V}$
Vandermonde matrix. And let multiply the above formula from the left. We get:

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = V \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix} = V^{-1}$$

The inverse Vandermonde matrix

..for the rescue

1. Recover the polynomial $p(x)$ from the point-value pairs

   $$\{(\gamma_0, p(\gamma_0)), (\gamma_1, p(\gamma_1)), \ldots, (\gamma_{n-1}, p(\gamma_{n-1}))\}$$

2. by doing a single matrix multiplication of V^{-1} by the vector $[y_0, y_1, \ldots, y_{n-1}]$.
3. Multiplying a vector with n entries with $n \times n$ matrix takes $O(n^2)$ time.
4. No benefit so far...

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What is the inverse of the Vandermonde matrix

Vandermonde matrix is famous, beautiful and well known – a celebrity matrix

Claim

$$V^{-1} = \frac{1}{n}\begin{pmatrix} 1 & \beta_0 & \beta_0^2 & \beta_0^3 & \cdots & \beta_0^{n-1} \\ 1 & \beta_1 & \beta_1^2 & \beta_1^3 & \cdots & \beta_1^{n-1} \\ 1 & \beta_2 & \beta_2^2 & \beta_2^3 & \cdots & \beta_2^{n-1} \\ 1 & \beta_3 & \beta_3^2 & \beta_3^3 & \cdots & \beta_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \beta_{n-1} & \beta_{n-1}^2 & \beta_{n-1}^3 & \cdots & \beta_{n-1}^{n-1} \end{pmatrix},$$

where $\beta_j = (\gamma_j(n))^{-1}$.

Proof

Consider the (u, v) entry in the matrix $C = V^{-1}V$.
We have

$$C_{u,v} = \sum_{j=0}^{n-1} \frac{(\beta_u)^j (\gamma_j)^v}{n}.$$

As $\gamma_j = (\gamma_1)^j$.Thus,

$$C_{u,v} = \sum_{j=0}^{n-1} \frac{(\beta_u)^j ((\gamma_1)^j)^v}{n} = \sum_{j=0}^{n-1} \frac{(\beta_u)^j ((\gamma_1)^v)^j}{n} = \sum_{j=0}^{n-1} \frac{(\beta_u \gamma_v)}{n}$$

Clearly, if $u = v$ then

$$C_{u,u} = \frac{1}{n} \sum^{n-1} (\beta_u \gamma_u)^j = \frac{1}{n} \sum^{n-1} (1)^j = \frac{n}{n} = 1.$$

Proof continued...

If $u \neq v$ then,

$$\beta_u \gamma_v = (\gamma_u)^{-1} \gamma_v = (\gamma_1)^{-u} \gamma_1^v = (\gamma_1)^{v-u} = \gamma_{v-u}.$$

And

$$C_{u,v} = \frac{1}{n} \sum_{j=0}^{n-1} (\gamma_{v-u})^j = \frac{1}{n} \cdot \frac{\gamma_{v-u}^n - 1}{\gamma_{v-u} - 1} = \frac{1}{n} \cdot \frac{1 - 1}{\gamma_{v-u} - 1} = 0,$$

Proved that the matrix C have ones on the diagonal and zero everywhere else. ■

Recap...

1. n point-value pairs $\{(\gamma_0, y_0), \ldots, (\gamma_{n-1}, y_{n-1})\}$: of polynomial $p(x) = \sum_{i=0}^{n-1} a_i x^i$ over nth roots of unity.
2. Recover coefficients of polynomial by multiplying $[y_0, y_1, \ldots, y_n]$ by V^{-1}:

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Recovering continued...

1. recover coefficients of $p(\cdot)$...
2. ... compute $W(\cdot)$ on n values: $\beta_0, \ldots, \beta_{n-1}$.
3. $\{\beta_0, \ldots, \beta_{n-1}\} = \{\gamma_0, \ldots, \gamma_{n-1}\}$.
4. Indeed $\beta_i^n = (\gamma_i^{-1})^n = (\gamma_i^n)^{-1} = 1^{-1} = 1$.
5. Apply the **FFTAlg** algorithm on $W(x)$ to compute $a_0, \ldots, a_{n-1}$.

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Result

Theorem

Given n *point-value pairs of a polynomial* $p(x)$ *of degree* $n-1$ *over the set of* n *powers of the* n*th roots of unity, we can recover the polynomial* $p(x)$ *in* $O(n \log n)$ *time.*

Theorem

Given two polynomials of degree n*, they can be multiplied in* $O(n \log n)$ *time.*

5.4: Convolutions

Convolutions

1. Two vectors: $A = [a_0, a_1, \ldots, a_n]$ and $B = [b_0, \ldots, b_n]$.
2. ***dot product*** $A \cdot B = \langle A, B \rangle = \sum_{i=0}^{n} a_i b_i$.
3. A_r: shifting of A by $n - r$ locations to the left
4. Padded with zeros:, $a_j = 0$ for $j \notin \{0, \ldots, n\}$).
5. $A_r = \left[a_{n-r}, a_{n+1-r}, a_{n+2-r}, \ldots, a_{2n-r} \right]$ where $a_j = 0$ if $j \notin \left[0, \ldots, n \right]$.
6. **Observation**: $A_n = A$.

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Example of shifting

Example

For $A = [3, 7, 9, 15]$, $n = 3$
$A_2 = [7, 9, 15, 0]$,
$A_5 = [0, 0, 3, 7]$.

Definition

Definition

Let $c_i = A_i \cdot B = \sum_{j=n-i}^{2n-i} a_j b_{j-n+i}$, for $i = 0, \ldots, 2n$. The vector $[c_0, \ldots, c_{2n}]$ is the ***convolution*** of A and B.

question

How to compute the convolution of two vectors of length n?

Convolution via multiplication polynomials

1. $p(x) = \sum_{i=0}^{n} \alpha_i x^i$, and $q(x) = \sum_{i=0}^{n} \beta_i x^i$.
2. Coefficient of x^i in $r(x) = p(x)q(x)$ is $d_i = \sum_{j=0}^{i} \alpha_j \beta_{i-j}$.
3. Want to compute $c_i = A_i \cdot B = \sum_{j=n-i}^{2n-i} a_j b_{j-n+i}$.
4. Set $\alpha_i = a_i$ and $\beta_l = b_{n-l-1}$.

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3. Want to compute $c_i = A_i \cdot B = \sum_{j=n-i}^{2n-i} a_j b_{j-n+i}$.
4. Set $\alpha_i = a_i$ and $\beta_l = b_{n-l-1}$.

Convolution via multiplication polynomials

1. $p(x) = \sum_{i=0}^{n} \alpha_i x^i$, and $q(x) = \sum_{i=0}^{n} \beta_i x^i$.
2. Coefficient of x^i in $r(x) = p(x)q(x)$ is $d_i = \sum_{j=0}^{i} \alpha_j \beta_{i-j}$.
3. Want to compute $c_i = A_i \cdot B = \sum_{j=n-i}^{2n-i} a_j b_{j-n+i}$.
4. Set $\alpha_i = a_i$ and $\beta_l = b_{n-l-1}$.

Convolution by example

1. Consider coefficient of x^2 in product of $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$.
2. Sum of the entries on the anti diagonal:

	a_0+	a_1x	$+a_2x^2$	$+a_3x^3$
b_0			$a_2b_0x^2$	
$+b_1x$		$a_1b_1x^2$		
$+b_2x^2$	$a_0b_2x^2$			
$+b_3x^3$				

3. entry in the ith row and jth column is a_ib_j.

Convolution by example

1. Consider coefficient of x^2 in product of $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$.
2. Sum of the entries on the anti diagonal:

	$a_0 +$	$a_1 x$	$+a_2 x^2$	$+a_3 x^3$
b_0			$a_2 b_0 x^2$	
$+b_1 x$		$a_1 b_1 x^2$		
$+b_2 x^2$	$a_0 b_2 x^2$			
$+b_3 x^3$				

3. entry in the ith row and jth column is $a_i b_j$.

Convolution by example

1. Consider coefficient of x^2 in product of $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$.
2. Sum of the entries on the anti diagonal:

	$a_0 +$	$a_1 x$	$+a_2 x^2$	$+a_3 x^3$
b_0			$a_2 b_0 x^2$	
$+b_1 x$		$a_1 b_1 x^2$		
$+b_2 x^2$	$a_0 b_2 x^2$			
$+b_3 x^3$				

3. entry in the ith row and jth column is $a_i b_j$.

Convolution

Theorem

Given two vectors $A = [a_0, a_1, \ldots, a_n]$*,* $B = [b_0, \ldots, b_n]$ *one can compute their convolution in* $O(n \log n)$ *time.*

Proof.

Let $p(x) = \sum_{i=0}^{n} a_{n-i} x^i$ and let $q(x) = \sum_{i=0}^{n} b_i x^i$. Compute $r(x) = p(x)q(x)$ in $O(n \log n)$ time using the convolution theorem. Let $c_0, \ldots, c_{2n}$ be the coefficients of $r(x)$. It is easy to verify, as described above, that $[c_0, \ldots, c_{2n}]$ is the convolution of A and B. $\square$

Notes

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