

Chapter 29

Entropy, Randomness, and Information

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“If only once - only once - no matter where, no matter before what audience - I could better the record of the great Rastelli and juggle with thirteen balls, instead of my usual twelve, I would feel that I had truly accomplished something for my country. But I am not getting any younger, and although I am still at the peak of my powers there are moments - why deny it? - when I begin to doubt - and there is a time limit on all of us.”

– Romain Gary, The talent scout..

29.1. Entropy

Definition 29.1.1. The **entropy** in bits of a discrete random variable X is given by

$$\mathbb{H}(X) = - \sum_x \mathbb{P}[X = x] \lg \mathbb{P}[X = x].$$

Equivalently, $\mathbb{H}(X) = \mathbb{E} \left[\lg \frac{1}{\mathbb{P}[X]} \right]$.

The **binary entropy** function $\mathbb{H}(p)$ for a random binary variable that is 1 with probability p , is $\mathbb{H}(p) = -p \lg p - (1-p) \lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$. See [Figure 29.1](#).

The function $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0, 1]$ and achieves its maximum at $1/2$. For a concrete example, consider $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$. Namely, a coin that has $3/4$ probably to be heads have higher amount of “randomness” in it than a coin that has probability $7/8$ for heads.

We have $\mathbb{H}'(p) = -\lg p + \lg(1-p) = \lg \frac{1-p}{p}$ and $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$. Thus, $\mathbb{H}''(p) \leq 0$, for all $p \in (0, 1)$, and the $\mathbb{H}(\cdot)$ is concave in this range. Also, $\mathbb{H}'(1/2) = 0$, which implies that $\mathbb{H}(1/2) = 1$ is a maximum of the binary entropy. Namely, a balanced coin has the largest amount of randomness in it.

Example 29.1.2. A random variable X that has probability $1/n$ to be i , for $i = 1, \dots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^n \frac{1}{n} \lg \frac{1}{n} = \lg n$.

Note, that the entropy is oblivious to the exact values that the random variable can have, and it is sensitive only to the probability distribution. Thus, a random variables that accepts $-1, +1$ with equal probability has the same entropy (i.e., 1) as a fair coin.

Lemma 29.1.3. *Let X and Y be two independent random variables, and let Z be the random variable (X, Y) . Then $\mathbb{H}(Z) = \mathbb{H}(X) + \mathbb{H}(Y)$.*

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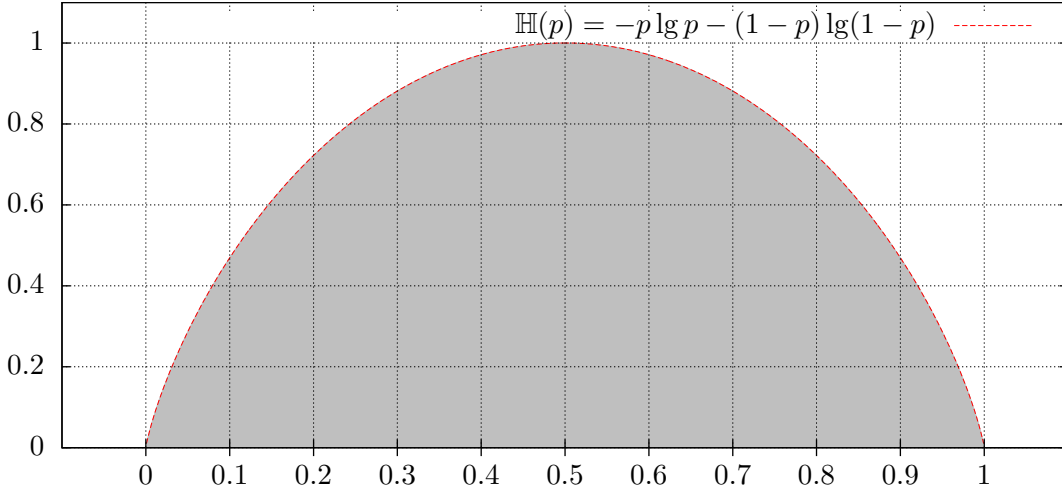


Figure 29.1: The binary entropy function.

Proof: In the following, summation are over all possible values that the variables can have. By the independence of X and Y we have

$$\begin{aligned}
\mathbb{H}(Z) &= \sum_{x,y} \mathbb{P}[(X,Y) = (x,y)] \lg \frac{1}{\mathbb{P}[(X,Y) = (x,y)]} \\
&= \sum_{x,y} \mathbb{P}[X = x] \mathbb{P}[Y = y] \lg \frac{1}{\mathbb{P}[X = x] \mathbb{P}[Y = y]} \\
&= \sum_x \sum_y \mathbb{P}[X = x] \mathbb{P}[Y = y] \lg \frac{1}{\mathbb{P}[X = x]} \\
&\quad + \sum_y \sum_x \mathbb{P}[X = x] \mathbb{P}[Y = y] \lg \frac{1}{\mathbb{P}[Y = y]} \\
&= \sum_x \mathbb{P}[X = x] \lg \frac{1}{\mathbb{P}[X = x]} + \sum_y \mathbb{P}[Y = y] \lg \frac{1}{\mathbb{P}[Y = y]} \\
&= \mathbb{H}(X) + \mathbb{H}(Y).
\end{aligned}$$

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Lemma 29.1.4. *Suppose that nq is integer in the range $[0, n]$. Then $\frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{nq} \leq 2^{n\mathbb{H}(q)}$.*

Proof: This trivially holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We know that

$$\binom{n}{nq} q^{nq} (1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

As such, since $q^{-nq} (1-q)^{-(1-q)n} = 2^{(-q \lg q - (1-q) \lg(1-q))n} = 2^{n\mathbb{H}(q)}$, we have

$$\binom{n}{nq} \leq q^{-nq} (1-q)^{-(1-q)n} = 2^{n\mathbb{H}(q)}.$$

As for the other direction, let $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$. We claim that $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq}$ is the largest term in $\sum_{k=0}^n \mu(k) = 1$. Indeed,

$$\Delta_k = \mu(k) - \mu(k+1) = \binom{n}{k} q^k (1-q)^{n-k} \left(1 - \frac{n-k}{k+1} \frac{q}{1-q} \right),$$

and the sign of this quantity is the sign of the last term, which is

$$\text{sign}(\Delta_k) = \text{sign} \left(1 - \frac{(n-k)q}{(k+1)(1-q)} \right) = \text{sign} \left(\frac{(k+1)(1-q) - (n-k)q}{(k+1)(1-q)} \right).$$

Now,

$$(k+1)(1-q) - (n-k)q = k+1 - kq - q - nq + kq = 1 + k - q - nq.$$

Namely, $\Delta_k \geq 0$ when $k \geq nq + q - 1$, and $\Delta_k < 0$ otherwise. Namely, $\mu(k) < \mu(k+1)$, for $k < nq$, and $\mu(k) \geq \mu(k+1)$ for $k \geq nq$. Namely, $\mu(nq)$ is the largest term in $\sum_{k=0}^n \mu(k) = 1$, and as such it is larger than the average. We have $\mu(nq) = \binom{n}{nq} q^{nq} (1-q)^{n-nq} \geq \frac{1}{n+1}$, which implies

$$\binom{n}{nq} \geq \frac{1}{n+1} q^{-nq} (1-q)^{-(n-nq)} = \frac{1}{n+1} 2^{n\mathbb{H}(q)}. \quad \blacksquare$$

Lemma 29.1.4 can be extended to handle non-integer values of q . This is straightforward, and we omit the easy but tedious details.

Corollary 29.1.5. *We have:*

- (i) $q \in [0, 1/2] \Rightarrow \binom{n}{\lfloor nq \rfloor} \leq 2^{n\mathbb{H}(q)}$.
- (ii) $q \in [1/2, 1] \Rightarrow \binom{n}{\lceil nq \rceil} \leq 2^{n\mathbb{H}(q)}$.
- (iii) $q \in [1/2, 1] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lfloor nq \rfloor}$.
- (iv) $q \in [0, 1/2] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lceil nq \rceil}$.

The bounds of **Lemma 29.1.4** and **Corollary 29.1.5** are loose but sufficient for our purposes. As a sanity check, consider the case when we generate a sequence of n bits using a coin with probability q for head, then by the Chernoff inequality, we will get roughly nq heads in this sequence. As such, the generated sequence Y belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences that have similar probability. As such, $\mathbb{H}(Y) \approx \lg \binom{n}{nq} = n\mathbb{H}(q)$, by **Example 29.1.2**, a fact that we already know from **Lemma 29.1.3**.

29.1.1. Extracting randomness

Entropy can be interpreted as the amount of unbiased random coin flips can be extracted from a random variable.

Definition 29.1.6. An extraction function *Ext* takes as input the value of a random variable X and outputs a sequence of bits y , such that $\mathbb{P}[\text{Ext}(X) = y \mid |y| = k] = \frac{1}{2^k}$, whenever $\mathbb{P}[|y| = k] > 0$, where $|y|$ denotes the length of y .

As a concrete (easy) example, consider X to be a uniform random integer variable out of $0, \dots, 7$. All that $\text{Ext}(X)$ has to do in this case, is to compute the binary representation of x . However, note that [Definition 29.1.6](#) is somewhat more subtle, as it requires that all extracted sequence of the same length would have the same probability.

Thus, for X a uniform random integer variable in the range $0, \dots, 11$, the function $\text{Ext}(x)$ can output the binary representation for x if $0 \leq x \leq 7$. However, what do we do if x is between 8 and 11? The idea is to output the binary representation of $x - 8$ as a two bit number. Clearly, [Definition 29.1.6](#) holds for this extraction function, since $\mathbb{P}[\text{Ext}(X) = 00 \mid |\text{Ext}(X)| = 2] = \frac{1}{4}$, as required. This scheme can be of course extracted for any range.

The following is obvious, but we provide a proof anyway.

Lemma 29.1.7. *Let x/y be a fraction, such that $x/y < 1$. Then, for any i , we have $x/y < (x+i)/(y+i)$.*

Proof: We need to prove that $x(y+i) - (x+i)y < 0$. The left side is equal to $i(x-y)$, but since $y > x$ (as $x/y < 1$), this quantity is negative, as required. \blacksquare

Theorem 29.1.8. *Suppose that the value of a random variable X is chosen uniformly at random from the integers $\{0, \dots, m-1\}$. Then there is an extraction function for X that outputs on average at least $\lceil \lg m \rceil - 1 = \lfloor \mathbb{H}(X) \rfloor - 1$ independent and unbiased bits.*

Proof: We represent m as a sum of unique powers of 2, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$. Thus, we decomposed $\{0, \dots, m-1\}$ into a disjoint union of blocks that have sizes which are distinct powers of 2. If a number falls inside such a block, we output its relative location in the block, using binary representation of the appropriate length (i.e., k if the block is of size 2^k). One can verify that this is an extraction function, fulfilling [Definition 29.1.6](#).

Now, observe that the claim holds trivially if m is a power of two. Thus, consider the case that m is not a power of 2. If X falls inside a block of size 2^k then the entropy is k . Thus, for the inductive proof, assume that are looking at the largest block in the decomposition, that is $m < 2^{k+1}$, and let $u = \lfloor \lg(m - 2^k) \rfloor < k$. There must be a block of size 2^u in the decomposition of m . Namely, we have two blocks that we know in the decomposition of m , of sizes 2^k and 2^u . Note, that these two blocks are the largest blocks in the decomposition of m . In particular, $2^k + 2 * 2^u > m$, implying that $2^{u+1} + 2^k - m > 0$.

Let Y be the random variable which is the number of bits output by the extractor algorithm.

By [Lemma 29.1.7](#), since $\frac{m-2^k}{m} < 1$, we have

$$\frac{m-2^k}{m} \leq \frac{m-2^k + (2^{u+1} + 2^k - m)}{m + (2^{u+1} + 2^k - m)} = \frac{2^{u+1}}{2^{u+1} + 2^k}.$$

Thus, by induction (we assume the claim holds for all integers smaller than m), we have

$$\begin{aligned} \mathbb{E}[Y] &\geq \frac{2^k}{m}k + \frac{m-2^k}{m} \left(\underbrace{\lfloor \lg(m-2^k) \rfloor}_u - 1 \right) = \frac{2^k}{m}k + \frac{m-2^k}{m} \underbrace{(k-k+u-1)}_{=0} \\ &= k + \frac{m-2^k}{m}(u-k-1) \\ &\geq k + \frac{2^{u+1}}{2^{u+1} + 2^k}(u-k-1) = k - \frac{2^{u+1}}{2^{u+1} + 2^k}(1+k-u), \end{aligned}$$

since $u - k - 1 \leq 0$ as $k > u$. If $u = k - 1$, then $\mathbb{E}[Y] \geq k - \frac{1}{2} \cdot 2 = k - 1$, as required. If $u = k - 2$ then $\mathbb{E}[Y] \geq k - \frac{1}{3} \cdot 3 = k - 1$. Finally, if $u < k - 2$ then

$$\mathbb{E}[Y] \geq k - \frac{2^{u+1}}{2^k}(1 + k - u) = k - \frac{k - u + 1}{2^{k-u-1}} = k - \frac{2 + (k - u - 1)}{2^{k-u-1}} \geq k - 1,$$

since $(2 + i)/2^i \leq 1$ for $i \geq 2$. ■

Bibliography