Chapter 14

Network Flow II - The Vengeance

By Sariel Har-Peled, November 28, 2018[1] Version: 0.11

14.1. Accountability

The comic in Figure 14.1 is by Jonathan Shewchuk and is referring to the Calvin and Hobbes comics.

People that do not know maximum flows: essentially everybody.

Average salary on earth < \$5,000

People that know maximum flow - most of them work in programming related jobs and make at least \$10,000 a year.

Salary of people that learned maximum flows: > \$10,000

Salary of people that did not learn maximum flows: < \$5,000

Salary of people that know Latin: 0 (unemployed).

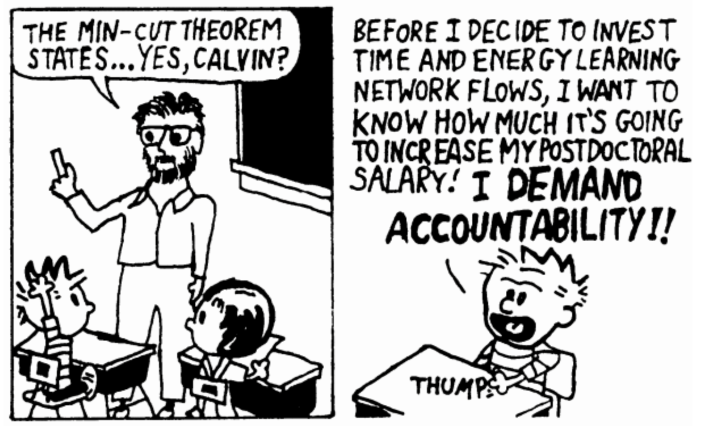


Figure 14.1: http://www.cs.berkeley.edu/~jrs/

Thus, by just learning maximum flows (and not knowing Latin) you can double your future salary!

14.2. The Ford-Fulkerson Method

The mtdFordFulkerson method is depicted on the right.

mtdFordFulkerson(G,s,t)
  Initialize flow f to zero
  while $\exists$ path π from s to t in G_f **do**
    $c_f(\pi) \leftarrow \min \left\{ c_f(u,v) \,\middle|\, (u,v) \in \pi \right\}$
    for $\forall (u,v) \in \pi$ **do**
      $f(u,v) \leftarrow f(u,v) + c_f(\pi)$
      $f(v,u) \leftarrow f(v,u) - c_f(\pi)$

Lemma 14.2.1. *If the capacities on the edges of G are integers, then mtdFordFulkerson runs in $O(m\,|f^*|)$ time, where $|f^*|$ is the amount of flow in the maximum flow and $m = |\mathsf{E}(\mathsf{G})|$.*

Proof: Observe that the mtdFordFulkerson method performs only subtraction, addition and min operations. Thus, if it finds an augmenting path π, then $c_f(\pi)$ must be a *positive* integer number. Namely, $c_f(\pi) \geq 1$. Thus, $|f^*|$ must be an integer number (by induction), and each iteration of the algorithm improves the flow by at least 1. It follows that after $|f^*|$ iterations the algorithm stops. Each iteration takes $O(m+n) = O(m)$ time, as can be easily verified. ■

The following observation is an easy consequence of our discussion.

Observation 14.2.2 (Integrality theorem). *If the capacity function c takes on only integral values, then the maximum flow f produced by the mtdFordFulkerson method has the property that $|f|$ is integer-valued. Moreover, for all vertices u and v, the value of $f(u, v)$ is also an integer.*

14.3. The Edmonds-Karp algorithm

The Edmonds-Karp algorithm works by modifying the mtdFordFulkerson method so that it always returns the shortest augmenting path in G_f (i.e., path with smallest number of edges). This is implemented by finding π using BFS in G_f.

Definition 14.3.1. For a flow f, let $\delta_f(v)$ be the length of the shortest path from the source s to v in the residual graph G_f. Each edge is considered to be of length 1.

We will shortly prove that, for any vertex $v \in V \setminus \{s, t\}$, the function $\delta_f(v)$, in the residual network G_f, increases monotonically with each flow augmentation. We delay proving this (key) technical fact (see Lemma 14.3.5 below), and first show its implications.

Lemma 14.3.2. *During the execution of the Edmonds-Karp algorithm, an edge (u, v) might disappear (and thus reappear) from G_f at most $n/2$ times throughout the execution of the algorithm, where $n = |\mathsf{V}(\mathsf{G})|$.*

Proof: Consider an iteration when the edge (u, v) disappears. Clearly, in this iteration the edge (u, v) appeared in the augmenting path π. Furthermore, this edge was fully utilized; namely, $c_f(\pi) = c_f(uv)$, where f is the flow in the beginning of the iteration when it disappeared. We continue running Edmonds-Karp till (u, v) "magically" reappears. This means that in the iteration before (u, v) reappeared in the residual graph, the algorithm handled an augmenting path σ that contained the *reverse* edge (v, u). Let g be the flow used to compute σ. We have, by the monotonicity of $\delta(\cdot)$ [i.e., Lemma 14.3.5 below], that

$$\delta_g(u) = \delta_g(v) + 1 \geq \delta_f(v) + 1 = \delta_f(u) + 2$$

as Edmonds-Karp is always augmenting along the shortest path. Namely, the distance of s to u had increased by 2 between its disappearance and its reappearance. Since $\delta_0(u) \geq 0$ and the maximum value of $\delta_?(u)$ is n, it follows that (u, v) can disappear and reappear at most $n/2$ times during the execution of the Edmonds-Karp algorithm. ■

Observe that $\delta_?(u)$ might become infinity at some point during the algorithm execution (i.e., u is no longer reachable from s). If so, by monotonicity, the edge (u, v) would never appear again, in the residual graph, in any future iteration of the algorithm.

Observation 14.3.3. *Every time we add an augmenting path during the execution of the Edmonds-Karp algorithm, at least one edge disappears from the residual graph $\mathsf{G}_?$. Indeed, every edge that realizes the residual capacity (of the augmenting path) will disappear once we push the maximum possible flow along this path.*

Lemma 14.3.4. *The Edmonds-Karp algorithm handles at most $O(nm)$ augmenting paths before it stops. Its running time is $O(nm^2)$, where $n = |\mathsf{V}(\mathsf{G})|$ and $m = |\mathsf{E}(\mathsf{G})|$.*

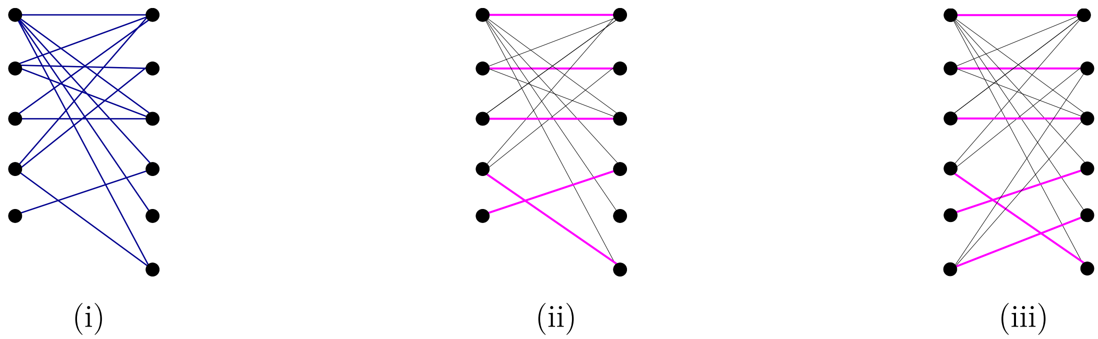


Figure 14.2: (i) A bipartite graph. (ii) A maximum matching in this graph. (iii) A perfect matching (in a different graph).

Proof: Every edge might disappear at most $n/2$ times during Edmonds-Karp execution, by Lemma 14.3.2. Thus, there are at most $nm/2$ edge disappearances during the execution of the Edmonds-Karp algorithm. At each iteration, we perform path augmentation, and at least one edge disappears along it from the residual graph. Thus, the Edmonds-Karp algorithm perform at most $O(mn)$ iterations.

Performing a single iteration of the algorithm boils down to computing an augmenting path. Computing such a path takes $O(m)$ time as we have to perform BFS to find the augmenting path. It follows, that the overall running time of the algorithm is $O(nm^2)$. ■

We still need to prove the aforementioned monotonicity property. (This is the only part in our discussion of network flow where the argument gets a bit tedious. So bear with us, after all, you are going to double your salary here.)

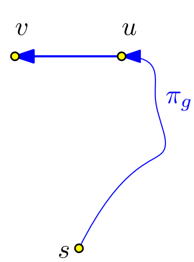
Lemma 14.3.5. *If the Edmonds-Karp algorithm is run on a flow network $\mathsf{G} = (V, E)$ with source s and sink t, then for all vertices $v \in V \setminus \{s, t\}$, the shortest path distance $\delta_f(v)$ in the residual network G_f increases monotonically with each flow augmentation.*

Proof: Assume, for the sake of contradiction, that this is false. Consider the flow just after the first iteration when this claim failed. Let f denote the flow before this (fatal) iteration was performed, and let g be the flow after.

Let v be the vertex such that $\delta_g(v)$ is minimal, among all vertices for which the monotonicity fails. Formally, this is the vertex v where $\delta_g(v)$ is minimal and

$$\delta_g(v) < \delta_f(v). \tag{*}$$

Let $\pi_g = s \to \cdots \to u \to v$ be the shortest path in G_g from s to v. Clearly, $(u, v) \in \mathsf{E}(\mathsf{G}_g)$, and thus $\delta_g(u) = \delta_g(v) - 1$.



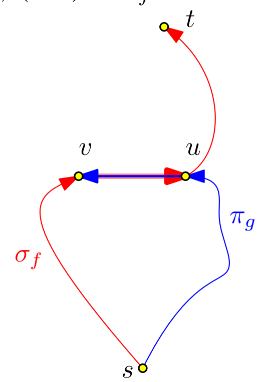
By the choice of v it must be that $\delta_g(u) \geq \delta_f(u)$, since otherwise the monotonicity property fails for u, and u is closer to s than v in G_g, and this, in turn, contradicts our choice of v as being the closest vertex to s that fails the monotonicity property. There are now two possibilities:

(i) If $(u, v) \in \mathsf{E}(\mathsf{G}_f)$ then

$$\delta_f(v) \le \delta_f(u) + 1 \le \delta_g(u) + 1 = \delta_g(v) - 1 + 1 = \delta_g(v).$$

This contradicts our assumptions that $\delta_f(v) > \delta_g(v)$.

(ii) If (u, v) is not in $\mathsf{E}(\mathsf{G}_f)$ then the augmenting path σ_f used in computing g from f contains the edge (v, u). Indeed, the edge (u, v) reappeared in the residual graph G_g (while not being present in G_f). The only way this can happens is if the augmenting path σ_f pushed a flow in the other direction on the edge (u, v). Namely, $(v, u) \in \sigma_f$.



However, the algorithm always augment along the shortest path. We have that

$$\delta_f(u) = \delta_f(v) + 1 \underbrace{>}_{(*)} \delta_g(v) + 1 > \delta_g(v) = \delta_g(u) + 1,$$

by the definition of u. Thus, $\delta_f(u) > \delta_g(u)$ (i.e., the monotonicity property fails for u) and $\delta_g(u) < \delta_g(v)$. A contradiction to the choice of v. ■

14.4. Applications and extensions for Network Flow

14.4.1. Maximum Bipartite Matching

Definition 14.4.1. For an undirected graph $\mathsf{G} = (V, E)$ a ***matching*** is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$, at most one edge of M is incident on v.

A ***maximum matching*** is a matching M such that for any matching M' we have $|M| \ge |M'|$.

A matching is ***perfect*** if it involves all vertices. See Figure 14.2 for examples of these definitions.

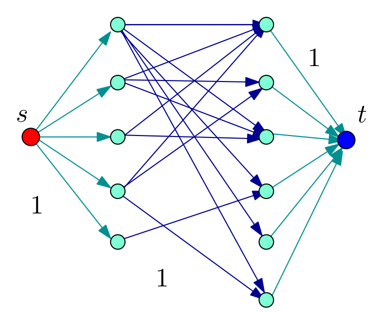


Figure 14.3

Theorem 14.4.2. *One can compute maximum bipartite matching using network flow in $O(nm)$ time, for a bipartite graph with n vertices and m edges.*

Proof: Given a bipartite graph G, we create a new graph with a new source on the left side and sink on the right, see Figure 14.3.

Direct all edges from left to right and set the capacity of all edges to 1. Let H be the resulting flow network. It is now easy to verify that by the Integrality theorem, a flow in H is either 0 or one on every edge, and thus a flow of value k in H is just a collection of k vertex disjoint paths between s and t in H, which corresponds to a matching in G of size k.

Similarly, given a matching of size k in G, it can be easily interpreted as realizing a flow in H of size k. Thus, computing a maximum flow in H results in computing a maximum matching in G. The running time of the algorithm is $O(nm^2)$. ■

14.4.2. Extension: Multiple Sources and Sinks

Given a flow network with several sources and sinks, how can we compute maximum flow on such a network?

The idea is to create a super source, that send all its flow to the old sources and similarly create a super sink that receives all the flow. See Figure 14.4. Clearly, computing flow in both networks in equivalent.

Bibliography

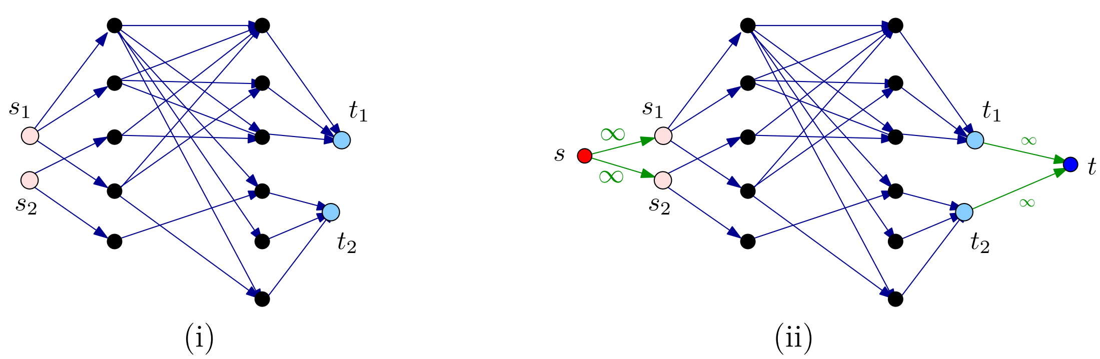


Figure 14.4: (i) A flow network with several sources and sinks, and (ii) an equivalent flow network with a single source and sink.