CS 473: Algorithms, Fall 2018

Approximation Algorithms III

Lecture 10

September 24, 2018

10.1: Subset Sum

Subset Sum

Subset Sum

Instance: $X = \{x_1, \ldots, x_n\}$ – n integer positive numbers, t - target number
Question: $\exists$ subset of X s.t. sum of its elements is t?

Assume $x_1, \ldots, x_n$ are all $\leq n$. Then this problem can be solved in

(A) The problem is still **NP-Hard**, so probably exponential time.

(B) $O(n^3)$.

(C) $2^{O(\log^2 n)}$.

(D) $O(n \log n)$.

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M: Max value input numbers.

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SolveSubsetSum (X, t, M)
  b[0 ... Mn] ⇐ false
    // b[x] is true if x can be
         // realized by subset of X.
  b[0] ← true.
  for i = 1, ..., n do
    for j = Mn down to x_i do
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R.T. $O(Mn^2)$

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Subset Sum

Efficient algorithm???

1. Algorithm solving **Subset Sum** in $O(Mn^2)$.
2. M might be prohibitly large...
3. if $M = 2^n \implies$ algorithm is not polynomial time.
4. **Subset Sum** is **NPC**.
5. Still want to solve quickly even if M huge.
6. Optimization version:

Subset Sum Optimization

Instance: (X, t): A set X of n positive integers, and a target number t.
Question: The largest number γ_{opt} one can represent as a subset sum of X which is smaller

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Subset Sum

2-approximation

Lemma

1. (X, t); *Given instance of* ***Subset Sum***. $\gamma_{\mathbf{opt}} \leq t$: *Opt.*
2. $\implies$ *Compute legal subset with sum* $\geq \gamma_{\mathbf{opt}}/2$.
3. *Running time* $O(n \log n)$.

Proof.

1. Sort numbers in X in decreasing order.
2. Greedily - add numbers from largest to smallest (if possible).
3. s: Generates sum.

Subset Sum

$\mathbf{2}$-approximation

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10.1.1: On the complexity of ε-approximation algorithms

Polynomial Time Approximation Schemes

Definition ( )

PROB: **Max**imization problem.
$\varepsilon > 0$: approximation parameter.
$\mathcal{A}(I, \varepsilon)$ is a ***polynomial time approximation scheme*** (**PTAS**) for **PROB**:

1. $\forall I$: $(1-\varepsilon)\left|\mathbf{opt}(I)\right| \leq \left|\mathcal{A}(I,\varepsilon)\right| \leq \left|\mathbf{opt}(I)\right|$,
2. $|\mathbf{opt}(I)|$: opt price,
3. $|\mathcal{A}(I,\varepsilon)|$: price of solution of $\mathcal{A}$.
4. $\mathcal{A}$ running time polynomial in n for fixed ε.

For minimization problem:
$|\mathbf{opt}(I)| \leq |\mathcal{A}(I,\varepsilon)| \leq (1+\varepsilon)|\mathbf{opt}(I)|$.

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Polynomial Time Approximation Schemes

1. Example: Approximation algorithm with running time $O(n^{1/\varepsilon})$ is a PTAS.
   Algorithm with running time $O(1/\varepsilon^n)$ is not.
2. Fully polynomial...

Definition ( )

An approximation algorithm is ***fully polynomial time approximation scheme*** (**FPTAS**) if it is a PTAS, and its running time is polynomial both in n and $1/\varepsilon$.

3. Example: PTAS with running time $O(n^{1/\varepsilon})$ is not a FPTAS.
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Approximating Subset Sum

Subset Sum Approx

Instance: (X, t, ε): A set X of n positive integers, a target number t, and parameter $\varepsilon > 0$.
Question: A number z that one can represent as a subset sum of X, such that $(1 - \varepsilon)\gamma_{\text{opt}} \leq z \leq \gamma_{\text{opt}} \leq t$.

Approximating Subset Sum

Looking again at the exact algorithm

ExactSubsetSum(**S**, t)
  $n \leftarrow |S|$
  $P_0 \leftarrow \{0\}$
  for $i = 1 \ldots n$ **do**
    $P_i \leftarrow P_{i-1} \cup (P_{i-1} + x_i)$
    `Remove from` P_i `all elements` $> t$

  return `largest element in` P_n

1. $S = \{a_1, \ldots, a_n\}$
   $x + S = \{a_1 + x, a_2 + x, \ldots a_n + x\}$
2. Lists might explode in size.

Trim the lists...

L': Inc. sorted list of numbers

```
Trim(L', δ)
   L = ⟨y_1 ... y_m⟩
   curr ← y_1
   L_out ← {y_1}
   for i = 2 ... m do
      if y_i > curr · (1 + δ)
         Append y_i to L_out
         curr ← y_i
   return L_out
```

Definition

For two positive real numbers $z \leq y$, the number y is a δ-approximation to z if $\frac{y}{1+\delta} \leq z \leq y$.

Observation

If $x \in L'$ then there exists a number $y \in L_{out}$ such that $y \leq x \leq y(1+\delta)$, where

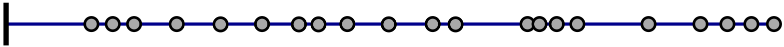
Trim the lists...

Trim(L', δ)
- $L = \langle y_1 \dots y_m \rangle$
- $curr \leftarrow y_1$
- $L_{out} \leftarrow \{y_1\}$
- **for** $i = 2 \dots m$ **do**
  - **if** $y_i > curr \cdot (1 + \delta)$
    - Append y_i to L_{out}
    - $curr \leftarrow y_i$
- **return** L_{out}

ApproxSubsetSum(**S**, t)
- // $S = \{x_1, \dots, x_n\}$,
- // $x_1 \leq x_2 \leq \dots \leq x_n$
- $n \leftarrow |S|$, $L_0 \leftarrow \{0\}$,
- $\delta = \varepsilon / 2n$
- **for** $i = 1 \dots n$ **do**
  - $E_i \leftarrow L_{i-1} \cup (L_{i-1} + x_i)$
  - $L_i \leftarrow$ **Trim**(E_i, δ)
  - Remove from L_i elems $> t$.
- **return** largest element in L_n

E_i: Computed by merging two sorted lists in linear time.

Understanding trimming



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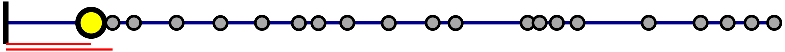

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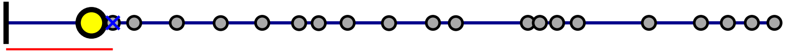

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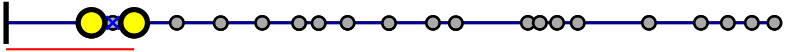

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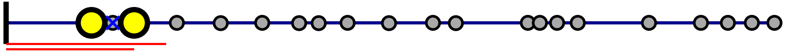

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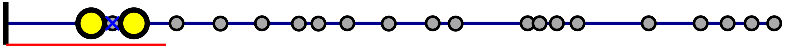

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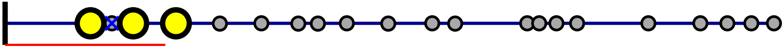

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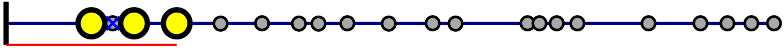

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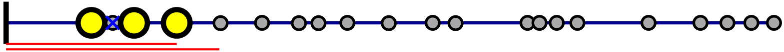

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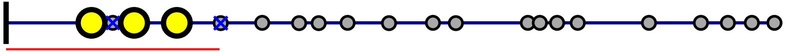

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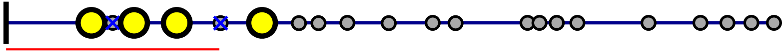

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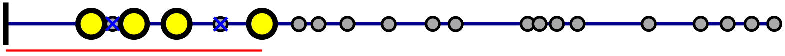

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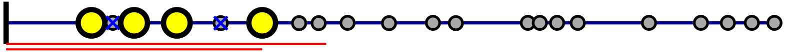

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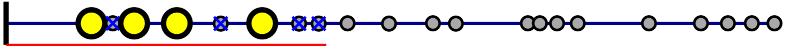

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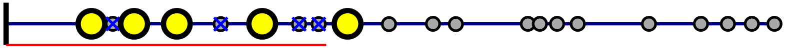

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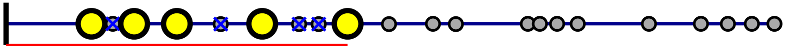

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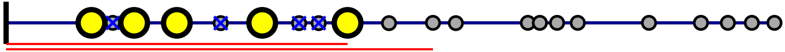

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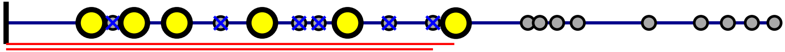

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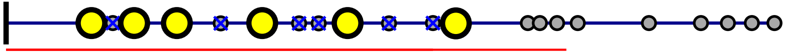

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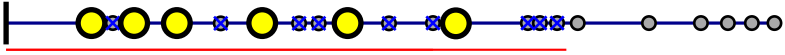

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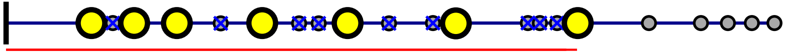

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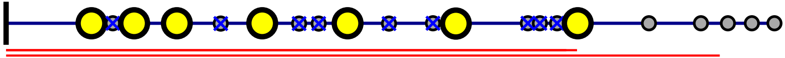

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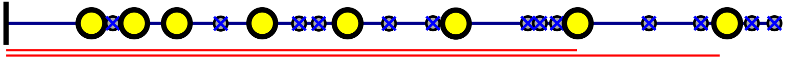

Remark

1. Can assume that trimmed lists L_i are sorted...
2. Algorithm: $E_i \leftarrow L_{i-1} \cup (L_{i-1} + x_i)$
3. So, this is just copy, shift, and merge of two sorted lists.
4. ... resulting in a sorted lest.
5. takes linear time in size of lists.

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Analysis

1. E_i list generated by algorithm in ith iteration.
2. P_i: list of numbers (no trimming).

Claim

For any $x \in P_i$ *there exists* $y \in L_i$ *such that* $y \leq x \leq (1+\delta)^i y$.

Proof

1. If $x \in P_1$ then follows by observation above.
2. If $x \in P_{i-1} \implies$ (induction) $\exists y' \in L_{i-1}$ s.t. $y' \leq x \leq (1+\delta)^{i-1} y'$.
3. By observation $\exists y \in L_i$ s.t. $y \leq y' \leq (1+\delta) y$, As such,

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Proof continued

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1. If $x \in P_i \setminus P_{i-1} \implies x = \alpha + x_i$, for some $\alpha \in P_{i-1}$.
2. By induction, $\exists \alpha' \in L_{i-1}$ s.t. $\alpha' \leq \alpha \leq (1+\delta)^{i-1}\alpha'$.
3. Thus, $\alpha' + x_i \in E_i$.
4. $\exists x' \in L_i$ s.t. $x' \leq \alpha' + x_i \leq (1+\delta)x'$.
5. Thus, $x' \leq \alpha' + x_i \leq \alpha + x_i = x \leq (1+\delta)^{i-1}\alpha' + x_i \leq (1+\delta)^{i-1}(\alpha' + x_i) \leq (1+\delta)^i x'$. ■

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Proof continued

Proof continued

1. If $x \in P_i \setminus P_{i-1} \implies x = \alpha + x_i$, for some $\alpha \in P_{i-1}$.
2. By induction, $\exists \alpha' \in L_{i-1}$ s.t. $\alpha' \leq \alpha \leq (1+\delta)^{i-1}\alpha'$.
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10.1.1.1: Running time

Running time of ApproxSubsetSum

Lemma

For $x \in [0, 1]$, it holds $\exp(x/2) \leq (1 + x)$.

Lemma

For $0 < \delta < 1$, and $x \geq 1$, we have

$$\log_{1+\delta} x \leq \frac{2 \ln x}{\delta} = O\left(\frac{\ln x}{\delta}\right).$$

See notes for a proof of lemmas.

Running time of ApproxSubsetSum

Observation

In a list generated by **Trim***, for any number x, there are no two numbers in the trimmed list between x and $(1+\delta)x$.*

Lemma

$|L_i| = O\Big((n/\varepsilon)\log n\Big)$*, for* $i = 1, \ldots, n$*.*

Running time of ApproxSubsetSum

Proof.

1. $L_{i-1} + x_i \subseteq [x_i, ix_i]$.
2. Trimming $L_{i-1} + x_i$ results in list of size

$$\log_{1+\delta} \frac{ix_i}{x_i} = O\left(\frac{\ln i}{\delta}\right) = O\left(\frac{\ln n}{\delta}\right),$$

3. Now, $\delta = \varepsilon/2n$, and

$$|L_i| \leq |L_{i-1}| + O\left(\frac{\ln n}{\delta}\right) \leq |L_{i-1}| + O\left(\frac{n \ln n}{\varepsilon}\right)$$

$$= O\left(\frac{n^2 \log n}{\right.$$

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$$= O\left(\frac{n^2 \log n}{}\right)$$

□

Running time of ApproxSubsetSum

Lemma

The running time of **ApproxSubsetSum** *is* $O\left(\frac{n^3}{\varepsilon}\log n\right)$.

Proof.

1. Running time of **ApproxSubsetSum** dominated by total length of $L_1, \ldots, L_n$.
2. Above lemma implies
$$\sum_i |L_i| = O\left(n \times \frac{n^2}{\varepsilon}\log n\right) = O\left(\frac{n^3}{\varepsilon}\log n\right)$$
.
3. **Trim** runs in time proportional to size of lists.
4. Overall R.T. $O\left(\frac{n^3}{\varepsilon}\log n\right)$

ApproxSubsetSum

Theorem

ApproxSubsetSum *returns* $u \leq t$*, s.t.*
$\frac{\gamma_{\text{opt}}}{1+\varepsilon} \leq u \leq \gamma_{\text{opt}} \leq t,$
γ_{opt}*: opt solution.*
Running time is $O((n^3/\varepsilon)\log n)$.

Proof.

1. Running time from above.
2. $\gamma_{\text{opt}} \in P_n$: optimal solution.
3. $\exists z \in L_n$, such that $z \leq \textbf{opt} \leq (1+\delta)^n z$
4. $(1+\delta)^n = (1+\varepsilon/2n)^n \leq \exp\left(\frac{\varepsilon}{2}\right) \leq 1+\varepsilon,$
   since $1 + x \leq e^x$ for $x \geq 0$.
5. $\gamma_{\text{opt}}/(1+\varepsilon) \leq z \leq \textbf{opt} \leq t$

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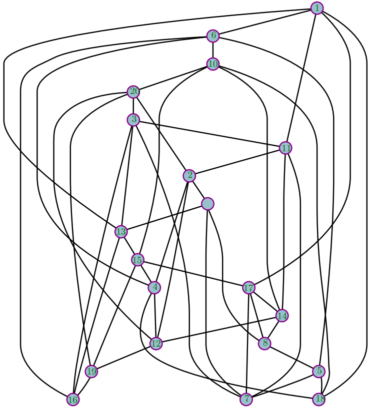
10.2: Maximal matching

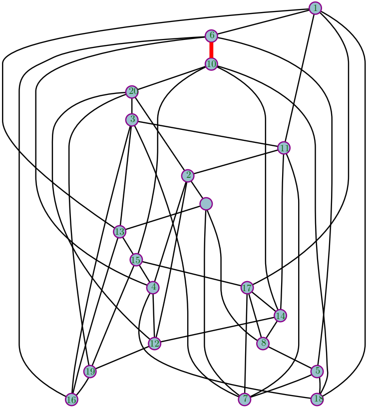
Maximal matching

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
2. Compute maximal matching...
3. $\boldsymbol{X} \subseteq \mathbf{E}$ which is maxim**al** and independent.
4. Maximal = can not improved by adding an edge.
5. Maximum = largest possible set among all possible sets.
6. Computing the maximum is hard then computing maximal solution.
7. Q: Find maximal matching quickly and of large size...

An example of the greedy algorithm...



An example of the greedy algorithm...



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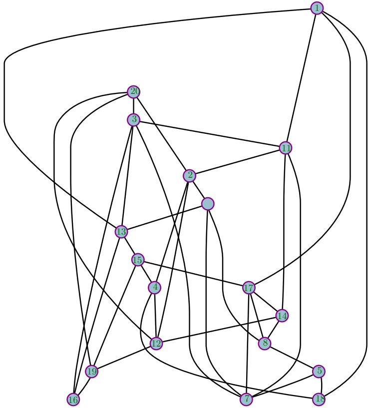

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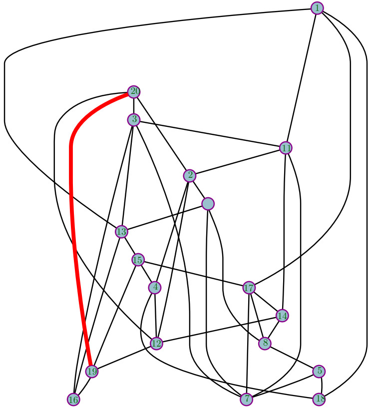

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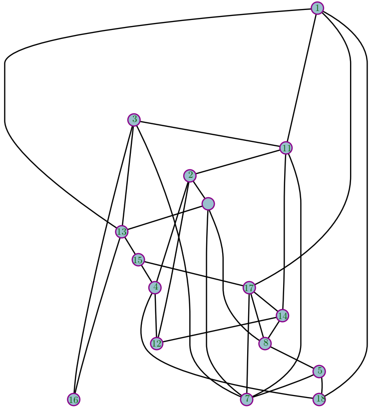

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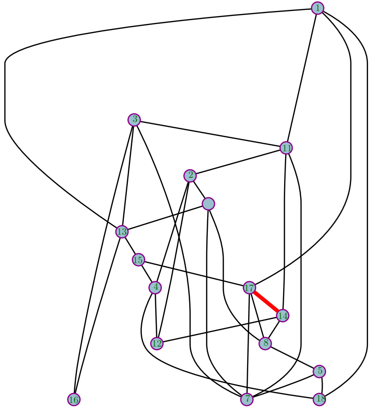

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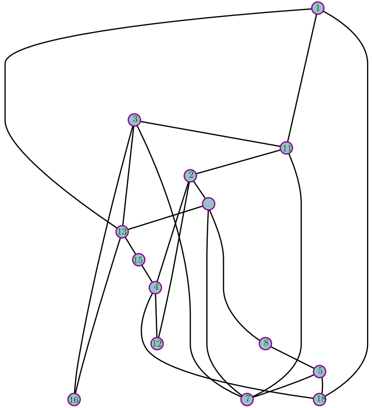

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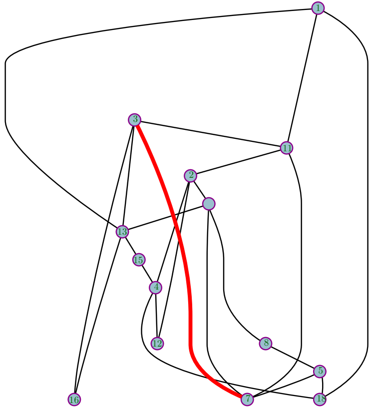

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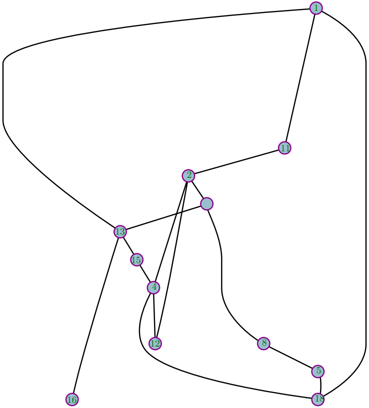

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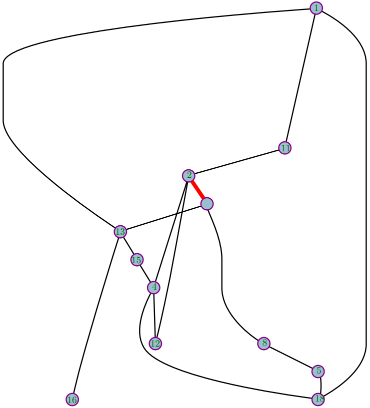

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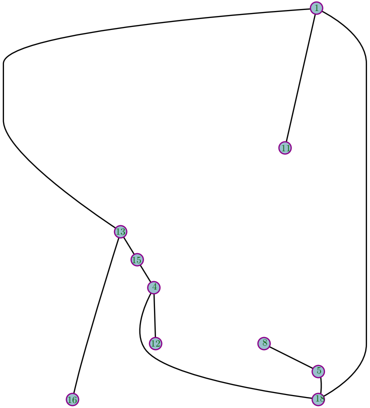

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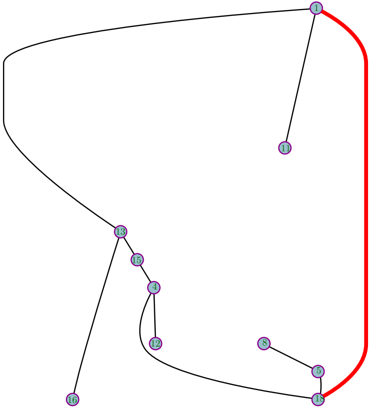

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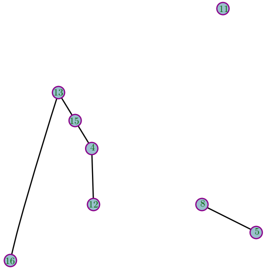

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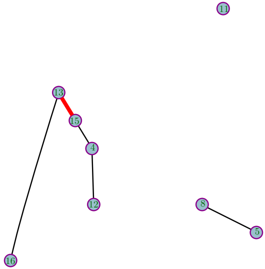

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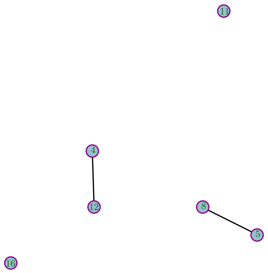

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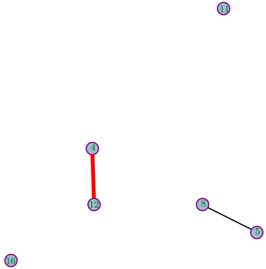

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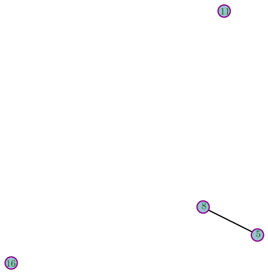

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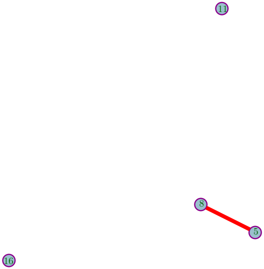

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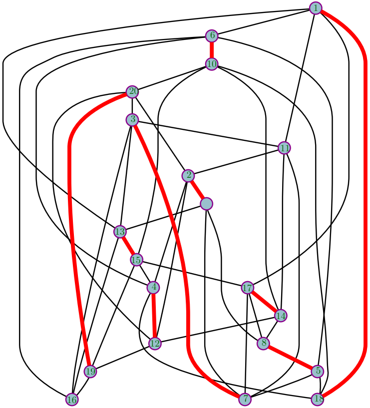

Maximal matching: Algorithm

1. Algorithm: Repeatedly pick an arbitrary edge and remove it.
2. M: Generated matching. X: Maximal matching.
3. Clearly a maximal matching...
4. This is a 2-approximation to the maximum matching.
5. Because...
6. Every edge in M "kills" two edges of X in the worst case.

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Maximal matching: Result

Theorem

Given a graph **G** *one can compute in* $O(n+m)$ *time, a maximal matching with at least* $|X|/2$ *edges, where* X *is the size of the maximum (optimal) matching.*

10.2.1: Bin packing

Bin packing

Problem definition

Bin Packing

Instance: v: Bin size. $S = \{\alpha_1, \ldots, \alpha_n\}$: n items
α_i: size of ith item.
Target: Find min # B, and a decomposition $S_1, \ldots, S_B$ of S, such that $\forall j \quad \sum_{x \in S_j} \leq v$.

1. $\cup_i S_i = S$ and $\forall i \neq j \quad S_i \cap S_j = \emptyset$.
2. **NP-Hard** from **Partition**.
3. **NP-Hard** to approximate within $3/2$.
4. Natural problem...
5. How to approximate?

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Bin packing: First fit

Analysis

Lemma

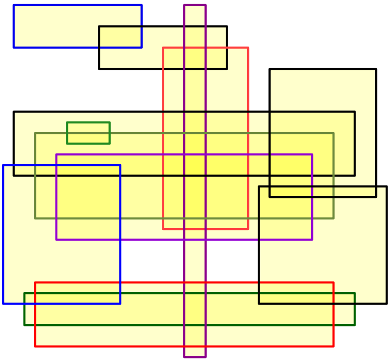
First fit is a $\mathbf{2}$-approximation.

Proof.

Observe that only one bin can have less than $\boldsymbol{v/2}$ content in it... $\square$

10.3: Independent set of axis-parallel rectangles

An example

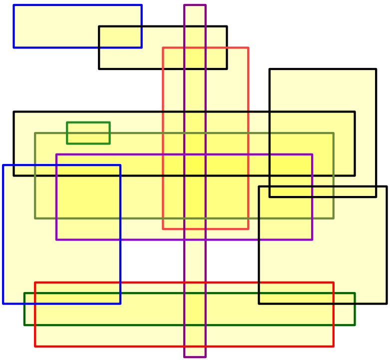


Input

Independent set of rectangles.

Assume: Open rectangles.

An example



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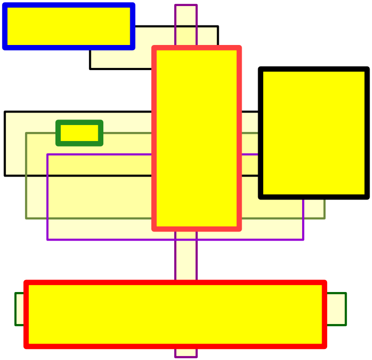


Independent set of rectangles.

Independent set of intervals

Clicker question

Given n intervals on the real line, computing the largest independent set of intervals on the real line, can be done in:

(A) $O(n)$ time.

(B) $O(n \log n)$ time.

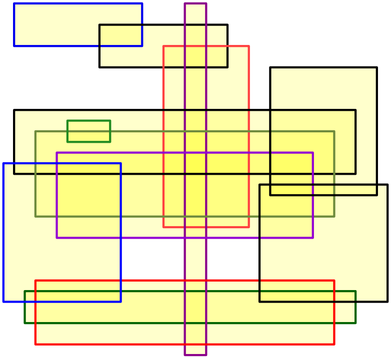
(C) $O(n^{3/2})$ time.

(D) $O(n^2)$ time.

(E) **NP-Hard**.

Independent set of rectangles

Algorithm: Divide & Conquer



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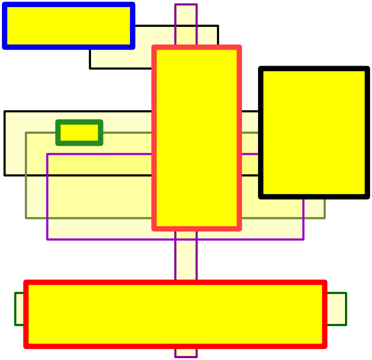

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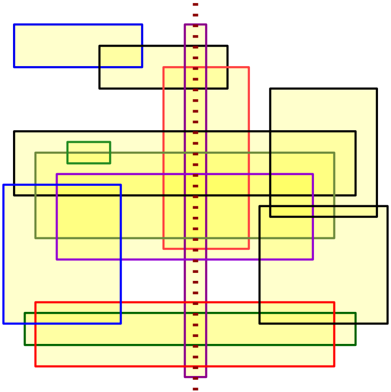

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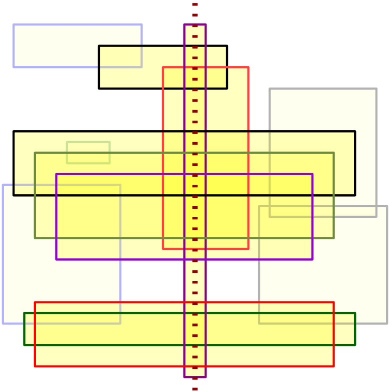

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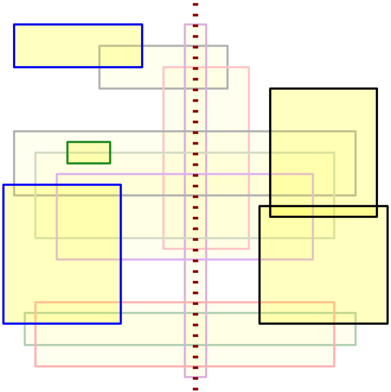

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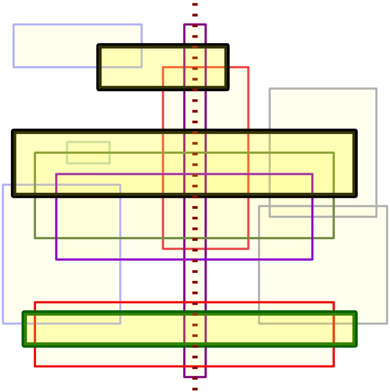

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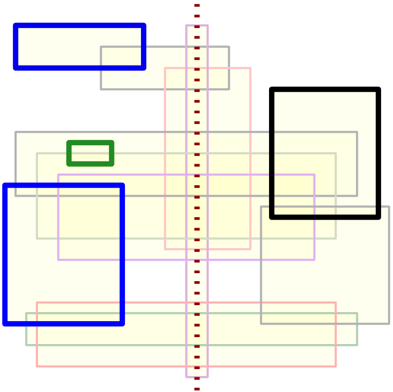

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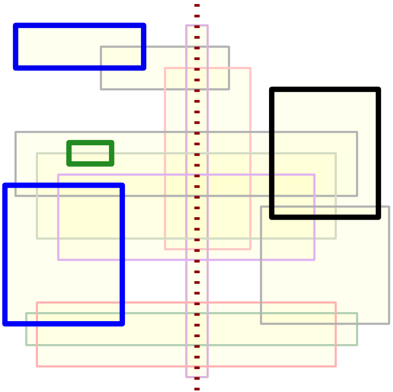

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$\mathcal{R}$: A set of axis parallel rectangles.

```
RectIndep(R):
  if |R| ≤ 10 then
    Solve by brute force
    return size of solution
  x_M: Median of right x-coordinate of rects in R
  ℓ: Vertical line through x_M.
  R_M: Rects of R intersecting ℓ
  R_L, R_R: Rectangles in R left/ right of ℓ.
  S_L ⇐ RectIndep(R_L)
  S_R ⇐ RectIndep(R_R)
  S_M ⇐ compute opt solution for R_M (intervals!)
```

Analysis

1. If $S_M \geq \mathrm{Opt}/(2\lg n)$... done.
2. $\mathrm{Opt}_L + \mathrm{Opt}_R \geq (1 - 1/(2\lg n))\mathrm{Opt}$.
3. By induction: $S_L \geq \mathrm{Opt}_L/(2\lg(n/2))$ and $S_R \geq \mathrm{Opt}_R/(2\lg(n/2))$.
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