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Linear time algorithms

Search and prune.

Divide and conquer.

Sep 11-1:58 PM

Median selection | K-order element.

$x_1 \; x_2 \; \cdots \; x_n$

K

Compute the k smallest numbers.

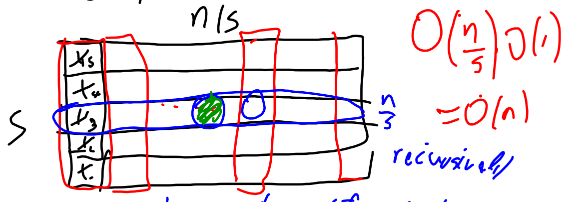
- Easy alg: Sort + return

$O(n \log n)$ $\quad \boxed{O(n)}$

Sep 11-2:02 PM

Median select $\quad k = \frac{n}{2}$

- $O(1)$ - Easy algorithm



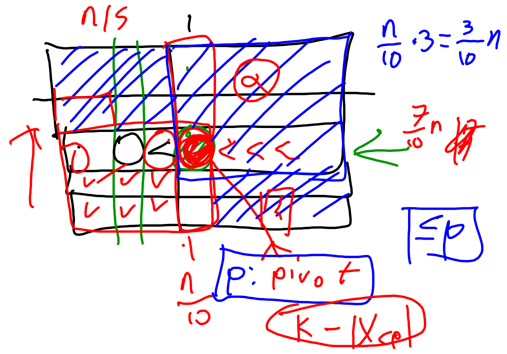
$O\left(\frac{n}{5}\right) O(1)$

$= O(n)$

recursively

Compute median of middle row.

Sep 11-2:04 PM



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Compute rank of p

$X = \{x_1, \ldots, x_n\}$

$X_{<p} = \{x_i \in X \mid x_i < p\}$

$X_{>p}$ $\quad |X_{<p}|$

$X = X_{<p} \cup \{p\} \cup X_{>p}$

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$|X_p| = k-1 \quad \text{rank}(p) = k \Rightarrow \text{done}$

$|X_{<p}| < k-1$

α: element of rank k of input

$\text{rank}(p) = |X_{<p}| + 1 < k$

$\Rightarrow p < \alpha$

Throw

Sep 11-2:14 PM

median (X[1..n], k)

Sort X[5i, ... 5i+4] for all i

Y ← [X[3], X[8] ... X[3+5i] ...]

p ← median(Y, $\frac{n}{10}$)

$X_{<p}$ = all element smaller than p

If $|X_p| = k-1$ then return p.

If $|X_p| < k-1$ then
 return median($X \setminus X_p$, $k-|X_p|$)

Else
 return median(X_p, k)

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Run Time

$$T(n) = O(n) + T\left(\frac{n}{5}\right) + T\left(\frac{7}{10}n\right)$$

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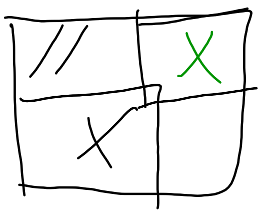

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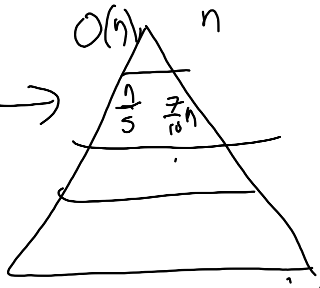
$\frac{p}{10}$ →



$\frac{1}{5} + \frac{7}{10} = \frac{9}{10} < 1$

5 3

$\left(\frac{9}{10}\right)^i n$

$$\sum_{i=0}^{\infty} O\left(\left(\frac{9}{10}\right)^i n\right) = O(n).$$

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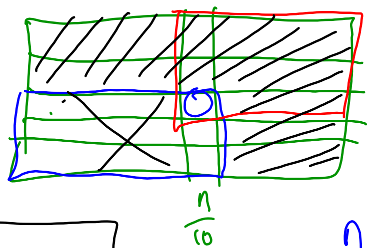

$\boxed{>p}$

$\frac{n}{10} \cdot 3$

$|X_{<p}| \geq \frac{3}{10}n$

$\frac{7}{10}n$

$|X \setminus X_{<p}| \leq n - \frac{3}{10}n$

Sep 11-2:28 PM

QuickSelect(X, k)

p ← X[rand(1, n)]

$X_{<p}$ = all number smaller than p

If $|X_p| = k-1$ then ret p

If $|X_p| < k$ then
 return QS($X \setminus X_p$, $k-|X_p|$)

else
 return QS($X_{<p}$, k)

E[RT] = O(n)

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$E[X \mid Y]$ conditional expectation

$E[X \mid Y = y]$

Lemma

$$E[E[X \mid Y]] = E[X]$$

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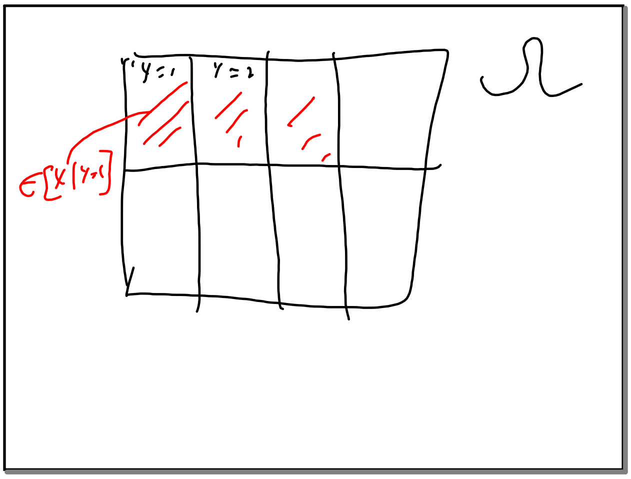

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Obs

Let S be a subproblem

$S \subseteq$ Input elements

$\alpha \in S$: The desired median

$m = |S|$

In the next level the recursive call is on a subproblem ~~of size~~ S'. We have that

$$E\big[|S'| \;\big|\; |S| = m\big] \le \frac{7}{8} m$$

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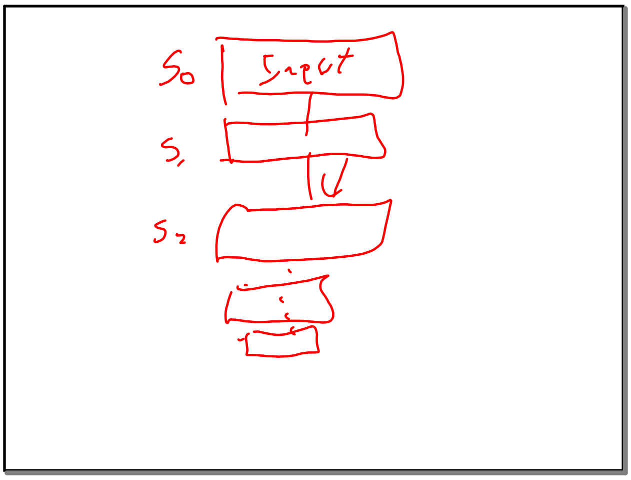

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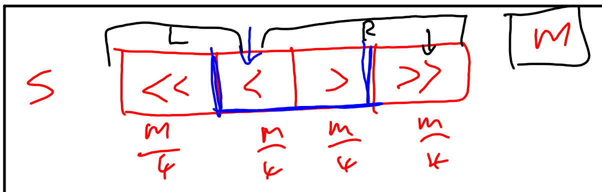

$|L|, |R| \le \frac{3}{4} m$ $\qquad \frac{1}{2}$

$$E\big[|S'| \;\big|\; |S|\big] \le \frac{1}{2} \cdot \frac{3}{4} m + \frac{1}{2} \cdot m = \frac{7}{8} m$$

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$Y_i \equiv$ RV size of the subproblem at level i of the recursion

$$E[Y_i \mid Y_{i-1}] \le \frac{7}{8} Y_{i-1}$$

$$E[Y_i] = E\big[E[Y_i \mid Y_{i-1}]\big] \le E\left[\frac{7}{8} Y_{i-1}\right] = \frac{7}{8} E[Y_{i-1}] = \left(\frac{7}{8}\right)^i n$$

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$$E[RT] = O\left[\sum_{i=} E[Y_i]\right]$$

$$T(n) = O(n) + T(n_{\text{sub problem}})$$

$$O\left(\sum_{i=1}^{\infty} \left(\frac{7}{8}\right)^i n\right) = O(n)$$

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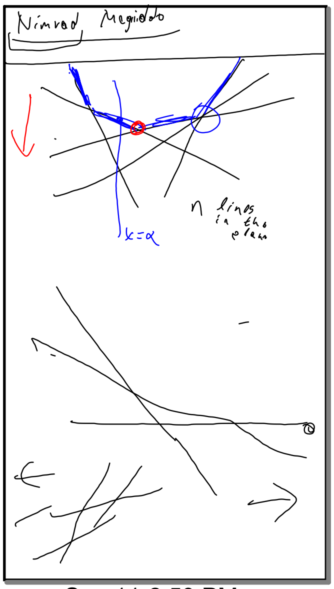

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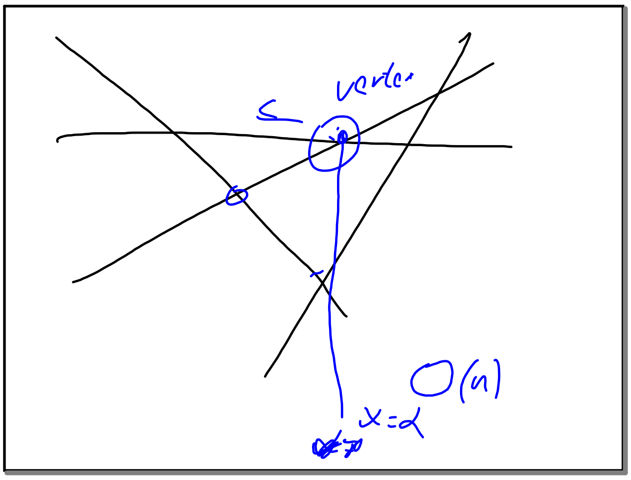

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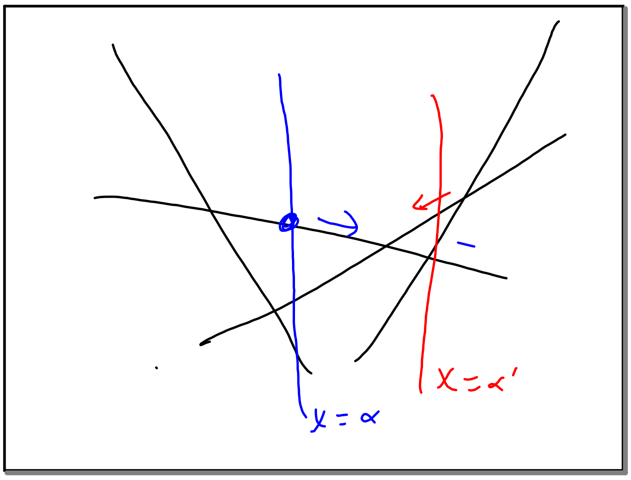

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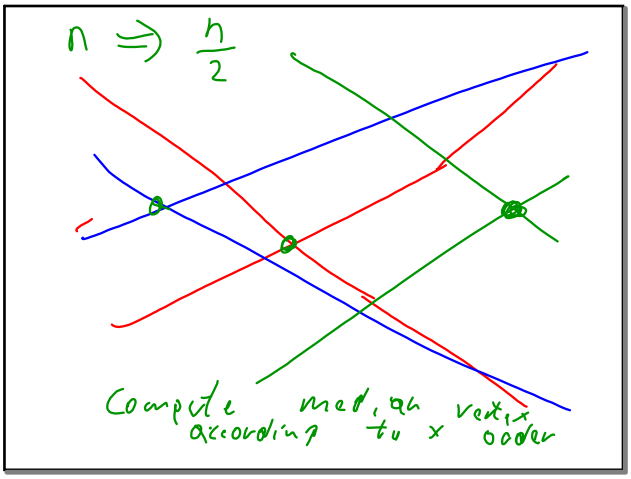

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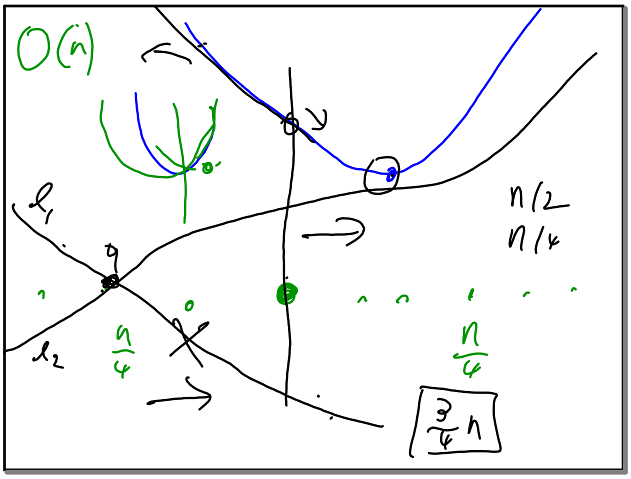

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$T(n) = O(n) + T(\frac{3}{4}n)$

$= O(n)$

2D Linear programming

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