# CS 473: Algorithms

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# LP Duality

Lecture 20

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# An easy LP?

$$\max cx \text{ subject to } Ax = b$$

which is compact form for

$$\begin{aligned} \max c_1x_1 + c_2x_2 + \ldots + c_nx_n & \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &= b_i \quad 1 \leq i \leq m \end{aligned}$$

**Question:** Is this a geneal LP problem or is it some how easy?

# An easy LP?

$$\max cx \text{ subject to } Ax = b$$

Basically reduces to linear system solving. Three cases for $Ax = b$.

- The system $Ax = b$ is infeasible, that is, no solution
- The system $Ax = b$ has a unique solution $x^*$ when $rank([A\ b]) = n$ (full rank). Optimum solution value is $cx^*$
- The system $Ax = b$ has infinite solutions when $rank([A\ b]) < n$. There all vectors of the form $x^* + y$ are feasible where $y$ is null-space$(A) = \{y \mid Ay = 0\}$. Let $d$ be dimension of null-space$(A)$ and let $e_1, e_2, \ldots, e_d$ be an orthonormal basis. Then $cx = cx^* + cy = cx^* + c(\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_d e_d$. If $ce_i \neq 0$ for any $i$ then optimum solution value is unbounded. Otherwise $cx^*$.

# LP Canonical Forms

Two basic canonical forms:

- $\mathbf{max}\ cx, Ax = b, x \geq \mathbf{0}$
- $\mathbf{max}\ cx, Ax \leq b, x \geq \mathbf{0}$

What makes LP non-trivial and different from linear system solving is the additional non-negativity constraint on variables.

# Part I

# Derivation and Definition of Dual LP

# Feasible Solutions and Lower Bounds

Consider the program

$$\begin{array}{lrrl} \text{maximize} & 4x_1 + & 2x_2 & \\ \text{subject to} & x_1 + & 3x_2 & \leq 5 \\ & 2x_1 - & 4x_2 & \leq 10 \\ & x_1 + & x_2 & \leq 7 \\ & x_1 & & \leq 5 \end{array}$$

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$$

1. $(0, 1)$ satisfies all the constraints and gives value $2$ for the objective function.
2. Thus, optimal value $\sigma^*$ is at least $4$.
3. $(2, 0)$ also feasible, and gives a better bound of $8$.
4. How good is $8$ when compared with $\sigma^*$?

# Obtaining Upper Bounds

1. Let us multiply the first constraint by **2** and the and add it to second constraint

$$\begin{array}{rrrl} 2( & x_1 + & 3x_2 & ) \le 2(5) \\ +1( & 2x_1 - & 4x_2 & ) \le 1(10) \\ \hline & 4x_1 + & 2x_2 & \le 20 \end{array}$$

2. Thus, 20 is an upper bound on the optimum value!

# Generalizing . . .

1. Multiply first equation by $y_1$, second by $y_2$, third by $y_3$ and fourth by $y_4$ (all of $y_1, y_2, y_3, y_4$ being positive) and add

$$\begin{array}{rrrl} y_1( & x_1 + & 3x_2 & ) \le y_1(5) \\ +y_2( & 2x_1 - & 4x_2 & ) \le y_2(10) \\ +y_3( & x_1 + & x_2 & ) \le y_3(7) \\ +y_4( & x_1 & & ) \le y_4(5) \\ \hline \multicolumn{4}{l}{(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 \le \ldots} \end{array}$$

2. $5y_1 + 10y_2 + 7y_3 + 5y_4$ is an upper bound, provided coefficients of $x_i$ are same as in the objective function, i.e.,

$$y_1 + 2y_2 + y_3 + y_4 = 4 \quad 3y_1 - 4y_2 + y_3 = 2$$

3. The best upper bound is when $5y_1 + 10y_2 + 7y_3 + 5y_4$ is minimized!

# Dual LP: Example

Thus, the optimum value of program

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leq 5 \\ & 2x_1 - 4x_2 \leq 10 \\ & x_1 + x_2 \leq 7 \\ & x_1 \leq 5 \end{aligned}$$

is upper bounded by the optimal value of the program

$$\begin{aligned} \text{minimize} \quad & 5y_1 + 10y_2 + 7y_3 + 5y_4 \\ \text{subject to} \quad & y_1 + 2y_2 + y_3 + y_4 = 4 \\ & 3y_1 - 4y_2 + y_3 = 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

# Dual Linear Program

Given a linear program $\Pi$ in canonical form

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots n \end{array}$$

the dual $\mathrm{Dual}(\Pi)$ is given by

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n} b_i y_i & \\ \text{subject to} & \sum_{i=1}^{n} y_i a_{ij} = c_j & j = 1, 2, \ldots d \\ & y_i \geq 0 & i = 1, 2, \ldots n \end{array}$$

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## Proposition

$\mathrm{Dual}(\mathrm{Dual}(\Pi))$ *is equivalent to* $\Pi$

# Duality Theorems

## Theorem (Weak Duality)

*If* $\mathbf{x}'$ *is a feasible solution to* $\mathbf{\Pi}$ *and* $\mathbf{y}'$ *is a feasible solution to* $\mathbf{Dual(\Pi)}$ *then* $\mathbf{c} \cdot \mathbf{x}' \leq \mathbf{y}' \cdot \mathbf{b}$.

# Duality Theorems

## Theorem (Weak Duality)

*If* $x'$ *is a feasible solution to* $\Pi$ *and* $y'$ *is a feasible solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x' \leq y' \cdot b$.

## Theorem (Strong Duality)

*If* $x^*$ *is an optimal solution to* $\Pi$ *and* $y^*$ *is an optimal solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x^* = y^* \cdot b$.

Many applications! Maxflow-Mincut theorem can be deduced from duality.

# Proof of Weak Duality

We already saw the proof by the way we derived it but we will do it again formally.

Since $y'$ is feasible to $\mathrm{Dual}(\Pi)$: $y'A = c$

Therefore $c \cdot x' = y'Ax'$

Since $x'$ is feasible $Ax' \leq b$ and hence,

$$c \cdot x' = y'Ax' \leq y' \cdot b$$

# Duality for another canonical form

$$\begin{aligned}
\text{maximize} \quad & 4x_1 + x_2 + 3x_3 \\
\text{subject to} \quad & x_1 + 4x_2 \leq 2 \\
& 2x_1 - x_2 + x_3 \leq 4 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

# Duality for another canonical form

$$\begin{array}{lrrrl} \text{maximize} & 4x_1+ & x_2+ & 3x_3 & \\ \text{subject to} & x_1+ & 4x_2 & & \leq 2 \\ & 2x_1- & x_2+ & x_3 & \leq 4 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

Choose non-negative $y_1, y_2$ and multiply inequalities

$$\begin{array}{lrrrl} \text{maximize} & 4x_1+ & x_2+ & 3x_3 & \\ \text{subject to} & y_1(x_1+ & 4x_2 & ) & \leq 2y_1 \\ & y_2(2x_1- & x_2+ & x_3) & \leq 4y_2 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

# Duality for another canonical form

Choose non-negative $y_1, y_2$ and multiply inequalities

$$\begin{array}{lrrrl} \text{maximize} & 4x_1 + & x_2 + & 3x_3 & \\ \text{subject to} & y_1(x_1 + & 4x_2 & ) & \leq 2y_1 \\ & y_2(2x_1 - & x_2 + & x_3) & \leq 4y_2 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

Adding the inequalities we get an inequality below that is valid for any feasible $x$ and any non-negative $y$:

$$(y_1 + 2y_2)x_1 + (4y_1 - y_2)x_2 + y_2 \leq 2y_1 + 4y_2$$

Suppose we choose $y_1, y_2$ such that
$y_1 + 2y_2 \geq 4$ and $4y_2 - y_2 \geq 1$ and $2y_1 \geq 3$
Then, since $x_1, x_2, x_3 \geq 0$, we have $4x_1 + x_2 + 3x_3 \leq 2y_1 + 4y_2$

# Duality for another canonical form

$$\begin{array}{lrrrl} \text{maximize} & 4x_1 + & x_2 + & 3x_3 & \\ \text{subject to} & x_1 + & 4x_2 & & \leq 2 \\ & 2x_1 - & x_2 + & x_3 & \leq 4 \\ & & & x_1, x_2, x_3 & \geq 0 \end{array}$$

is upper bounded by

$$\begin{array}{lrrl} \text{minimize} & 2y_1 + & 4y_2 & \\ \text{subject to} & y_1 + & 2y_2 & \geq 4 \\ & 4y_1 - & y_2 & \geq 1 \\ & 2y_1 & & \geq 3 \\ & & y_1, y_2 & \geq 0 \end{array}$$

# Duality for another canonical form

Compactly,

For the primal LP max $\boldsymbol{cx}$ subject to $\boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$
the dual LP is min $\boldsymbol{yb}$ subject to $\boldsymbol{yA} \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}$

# Some Useful Duality Properties

Assume primal LP is a maximization LP.

- For a given LP, Dual is another LP. The variables in the dual correspond to "non-trival" primal constraints and vice-versa.
- Dual of the dual LP give us back the primal LP.
- Weak and strong duality theorems.
- If primal is unbounded (objective achieves infinity) then dual LP is infeasible. Why? If dual LP had a feasible solution it would upper bound the primal LP which is not possible.
- If primal is infeasible then dual LP is unbounded.
- Primal and dual optimum solutions satisfy complementary slackness conditions (discussed soon).

# Part II

# Examples of Duality

# Max matching in bipartite graph as LP

Input: $G = (V = L \cup R, E)$

$$
\begin{aligned}
\textbf{max} \quad & \sum_{uv \in E} x_{uv} & \\
s.t. \quad & \sum_{uv \in E} x_{uv} \leq 1 & \forall v \in V. \\
& x_{uv} \geq 0 & \forall uv \in E
\end{aligned}
$$

# Network flow

$s$-$t$ flow in directed graph $G = (V, E)$ with capacities $c$. Assume for simplicity that no incoming edges into $s$.

$$
\begin{aligned}
\textbf{max} \quad & \sum_{(s,v)\in E} x(s,v) & \\
& \sum_{(u,v)\in E} x(u,v) - \sum_{(v,w)\in E} x(v,w) = 0 & \forall v \in V \setminus \{s,t\} \\
& x(u,v) \le c(u,v) & \forall (u,v) \in E \\
& x(u,v) \ge 0 & \forall (u,v) \in E.
\end{aligned}
$$

# Dual of Network Flow

# Part III

# Farkas Lemma and Strong Duality

# Optimization vs Feasibility

Suppose we want to solve LP of the form:

$$\textbf{max } \boldsymbol{c}\boldsymbol{x} \text{ subject to } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

It is an optimization problem. Can we reduce it to a decision problem?

# Optimization vs Feasibility

Suppose we want to solve LP of the form:

$$\max cx \text{ subject to } Ax \leq b$$

It is an optimization problem. Can we reduce it to a decision problem? Yes, via binary search. Find the largest values of $\sigma$ such that the system of inequalities

$$Ax \leq b, cx \geq \sigma$$

is *feasible*. Feasible implies that there is at least one solution.
Caveat: to do binary search need to know the range of numbers. Skip for now since we need to worry about precision issues etc.

# Certificate for (in)feasibility

Suppose we have a system of $m$ inequalities in $n$ variables defined by

$$Ax \leq b$$

- How can we convince some one that there is a feasible solution?
- How can we convince some one that there is **no** feasible solution?

# Theorem of the Alternatives

## Theorem

*Let $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$ and $\boldsymbol{b} \in \mathbb{R}^{\boldsymbol{m}}$. The system $\boldsymbol{Ax} \leq \boldsymbol{b}$ is either feasible of if it is infeasible then there is a $\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{m}}$ such that $\boldsymbol{y} \geq \mathbf{0}$ and $\boldsymbol{yA} = \mathbf{0}$ and $\boldsymbol{yb} < \mathbf{0}$.*

In other words, if $\boldsymbol{Ax} \leq \boldsymbol{b}$ is infeasible we can demonstrate it via the following compact contradiction. Find a non-negative combination of the rows of $\boldsymbol{A}$ (given by certificate $\boldsymbol{y}$) to derive $\mathbf{0} < \mathbf{1}$.

The preceding theorem can be used to prove strong duality. A fair amount of formal detail though geometric intuition is reasonable.

# Farkas Lemma

From the theorem of alternatives we can derive a useful version of Farkas lemma.

**Theorem**

*A system* $Ax = b, x \geq \mathbf{0}$ *is either feasible or there is a* $y$ *such that* $yA \geq \mathbf{0}$ *and* $yb < \mathbf{0}$. *Then the following hold:*

Nice geometric interpretation.

# Complementary Slackness

## Theorem

*Let $x^*$ be any optimum solution to primal LP $\Pi$ in the canonical form $\max cx, Ax \leq b, x \geq 0$ and $y^*$ be an optimum solution to the dual LP $Dual(\Pi)$ which is $\min yb, yA \geq c, y \geq 0$. Then the following hold.*

- *If $y_i^* > 0$ then $\sum_{j=1}^{n} a_{ij}x_j = b_j$ (the primal constraint for row $i$ is tight).*
- *If $x_j^* > 0$ then $\sum_{i=1}^{m} y_i a_{ij} = c_j$ (the dual constraint for row $j$ is tight).*

*The converse also hold: if $x^*$ and $y^*$ are primal and dual feasible and satisfy complementary slackness conditions then both must be optimal.*

Very useful in various applications. Nice geometric interpretation.

# Part IV

# Integer Linear Programming

# Integer Linear Programming

## Problem

Find a vector $x \in Z^d$ (integer values) that

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots n \end{array}$$

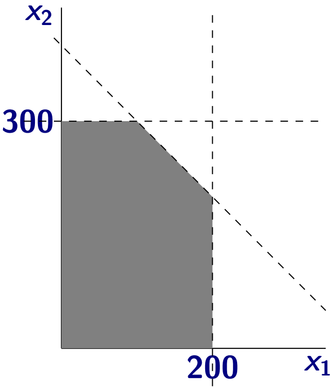
Input is matrix $A = (a_{ij}) \in \mathbb{R}^{n \times d}$, column vector $b = (b_i) \in \mathbb{R}^n$, and row vector $c = (c_j) \in \mathbb{R}^d$

# Factory Example

$$\begin{array}{llll} \text{maximize} & & x_1 + 6x_2 & \\ \text{subject to} & x_1 \leq 200 & x_2 \leq 300 & x_1 + x_2 \leq 400 \\ & & x_1, x_2 \geq 0 & \end{array}$$

Suppose we want $x_1, x_2$ to be integer valued.

# Factory Example Figure




1. Feasible values of $x_1$ and $x_2$ are integer points in shaded region
2. Optimization function is a line; moving the line until it just leaves the final integer point in feasible region, gives optimal values

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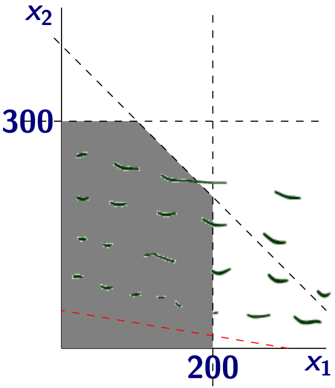


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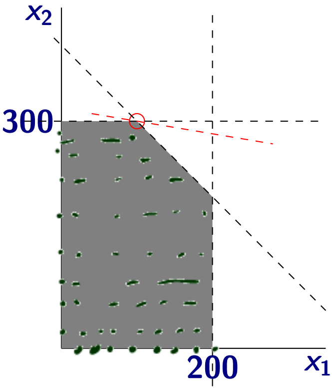


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# Integer Programming

Can model many difficult discrete optimization problems as integer programs!

Therefore integer programming is a hard problem. NP-hard.

Can relax integer program to linear program and *approximate*.

Practice: integer programs are solved by a variety of methods

1. branch and bound
2. branch and cut
3. adding cutting planes
4. linear programming plays a fundamental role

# Example: Maximum Independent Set

## Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in $S$. That is, if $u, v \in S$ then $(u, v) \notin E$.

**Input** Graph $G = (V, E)$

**Goal** Find maximum sized independent set in $G$

$x_u \quad u \in V \quad x_u \in \{0,1\}$

$$\max \sum_{u \in V} x_u$$

$$x_u + x_v \leq 1 \quad \forall uv \in E$$

$$x_u \in \{0,1\} \quad \forall u \in V$$

$$0 \leq x_u \leq 1 \quad x_u \in \mathbb{Z}$$

# Example: Dominating Set

## Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is a dominating set if for all $v \in V$, either $v \in S$ or a neighbor of $v$ is in $S$.

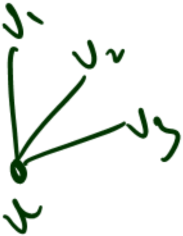
**Input** Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

**Goal** Find minimum weight dominating set in $G$

$$x_u \in \{0,1\} \quad u \in V$$

$$\min \sum_{u \in V} x_u$$

$$x_u + \sum_{v \in N(u)} x_v \geq 1 \quad \forall u \in V$$




$$x_u \in \{0,1\} \quad \forall u \in V$$

# Example: s-t minimum cut and implicit constraints

**Input** Graph $G = (V, E)$, edge capacities $c(e), e \in E$. $s, t \in V$

**Goal** Find minimum capacity $s$-$t$ cut in $G$.

$x_e \quad e \in E \qquad x_e \in \{0,1\}$

$$\min \sum_e c(e)\, x(e)$$

$$\sum_{e \in p} x_e \geq 1 \qquad \forall p \in \mathcal{P}_{s,t}$$

$$x_e \in [0,1]$$

$x_v \in \{0,1\} \quad v \in V$

$x_v = 0 \Rightarrow v$ is on $s$-side

$= 1 \Rightarrow v$ is on $t$-side

---

$y_e \in \{0,1\}$ is $e$ cut or not

$$\min \sum_e c(e)\, y(e)$$

$$x_s = 0, \; x_t = 1$$

$0 \leq y_e \leq 1 \qquad y_{uv} \geq x_v - x_u$

$0 \leq x_u \leq 1 \qquad x_u \in \{0,1\},\; y_e \in \{0,1\}$

# Linear Programs with Integer Vertices

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# Linear Programs with Integer Vertices

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*Luck* or *Structure*:

1. Linear program for flows with integer capacities have integer vertices
2. Linear program for matchings in bipartite graphs have integer vertices
3. A complicated linear program for matchings in general graphs have integer vertices.

All of above problems can hence be solved efficiently.

# Linear Programs with Integer Vertices

Meta Theorem: A combinatorial optimization problem can be solved efficiently if and only if there is a linear program for problem with integer vertices.

Consequence of the Ellipsoid method for solving linear programming.

*In a sense* linear programming and other geometric generalizations such as convex programming are the most general problems that we can solve efficiently.

# Summary

1. Linear Programming is a useful and powerful (modeling) problem.
2. Can be solved in polynomial time. Practical solvers available commercially as well as in open source. Whether there is a strongly polynomial time algorithm is a major open problem.
3. Geometry and linear algebra are important to understand the structure of LP and in algorithm design. Vertex solutions imply that LPs have poly-sized optimum solutions. This implies that LP is in **NP**.
4. Duality is a critical tool in the theory of linear programming. Duality implies the Linear Programming is in **co-NP**. Do you see why?
5. Integer Programming in **NP-Complete**. LP-based techniques critical in heuristically solving integer programs.