Network Flow Algorithms

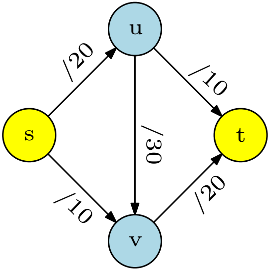
Lecture 14

October 12, 2016

Part I

Algorithm(s) for Maximum Flow

Greedy Approach



1. Begin with $f(e) = 0$ for each edge.
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$.
3. **Augment** flow along this path.
4. Repeat augmentation for as long as possible.

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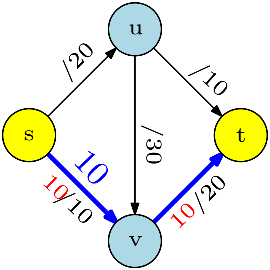

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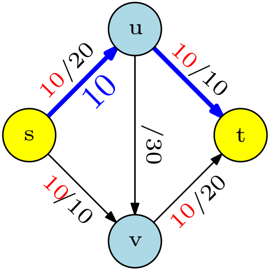

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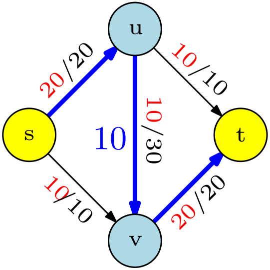

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Greedy Approach: Issues

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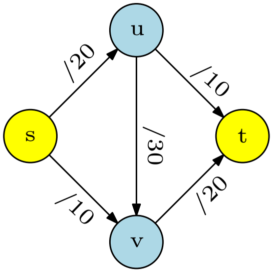

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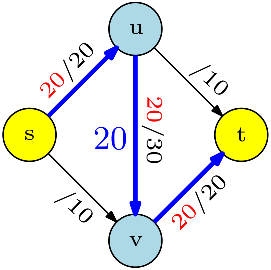

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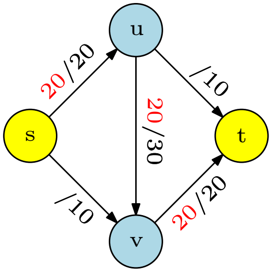

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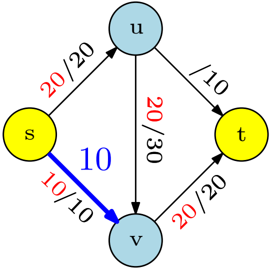

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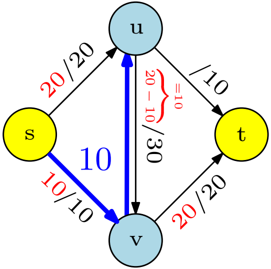

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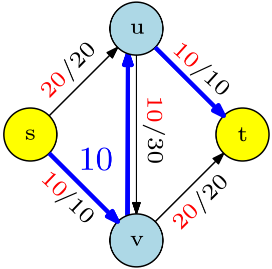

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Residual Graph

The "leftover" graph

Definition

For a network $G = (V, E)$ and flow f, the **residual graph** $G_f = (V', E')$ of G with respect to f is

1. $V' = V$,
2. **Forward Edges**: For each edge $e \in E$ with $f(e) < c(e)$, we add $e \in E'$ with capacity $c(e) - f(e)$.
3. **Backward Edges**: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we add $(v, u) \in E'$ with capacity $f(e)$.

Residual Graph Example

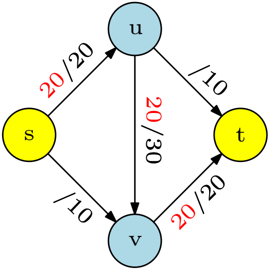

Figure: Flow on edges is indicated in red

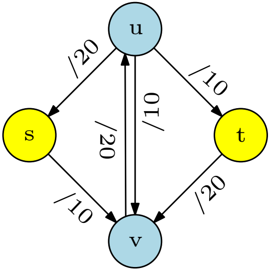

Figure: Residual Graph

Residual graph has...

Given a network with n vertices and m edges, and a valid flow f in it, the residual network G_f, has

(A) m edges.

(B) $\leq 2m$ edges.

(C) $\leq 2m + n$ edges.

(D) $4m + 2n$ edges.

(E) nm edges.

(F) just the right number of edges - not too many, not too few.

Residual Graph Property

Observation: Residual graph captures the "residual" problem exactly.

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Lemma

Let f and f' be two flows in G with $v(f') \geq v(f)$. Then there is a flow f'' of value $v(f') - v(f)$ in G_f.

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Definition of $+$ and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

```
MaxFlow(G, s, t):
    if the flow from s to t is 0 then
        return 0
    Find any flow f with v(f) > 0 in G
    Recursively compute a maximum flow f' in G_f
    Output the flow f + f'
```

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Iterative algorithm for finding a maximum flow:

```
MaxFlow(G, s, t):
    Start with flow f that is 0 on all edges
    while there is a flow f' in G_f with v(f') > 0 do
        f = f + f'
        Update G_f

    Output f
```

Ford-Fulkerson Algorithm

algFordFulkerson

```
for every edge e, f(e) = 0
G_f is residual graph of G with respect to f
while G_f has a simple s-t path do
    let P be simple s-t path in G_f
    f = augment(f, P)
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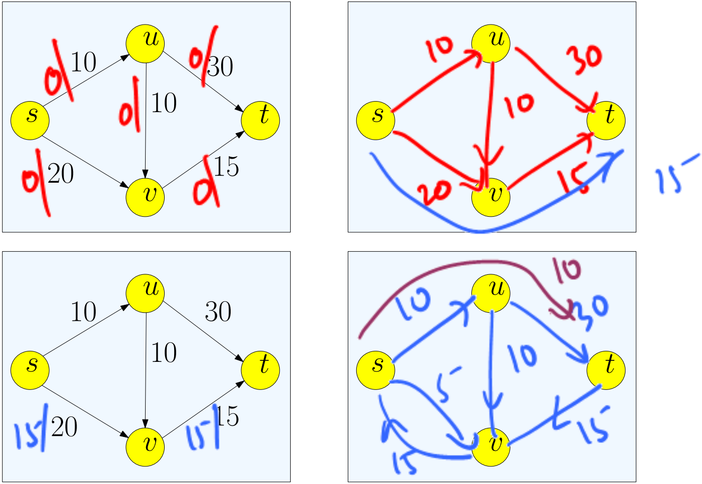
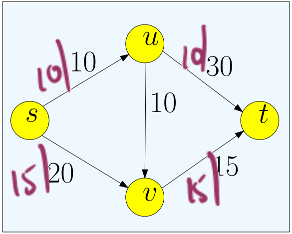
augment(*f*, *P*)

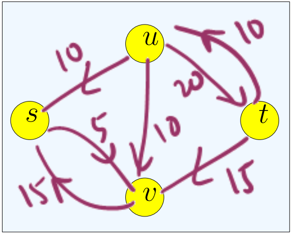
```
let b be bottleneck capacity,
    i.e., min capacity of edges in P (in G_f)
for each edge (u, v) in P do
    if e = (u, v) is a forward edge then
        f(e) = f(e) + b
    else (* (u, v) is a backward edge *)
        let e = (v, u) (* (v, u) is in G *)
        f(e) = f(e) - b
return f
```

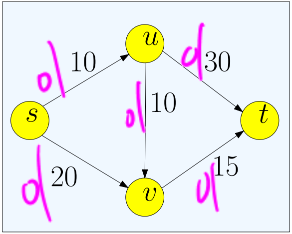
Example

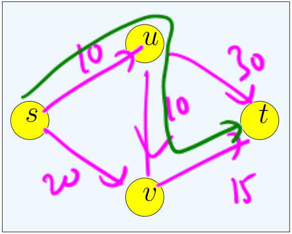


Example continued









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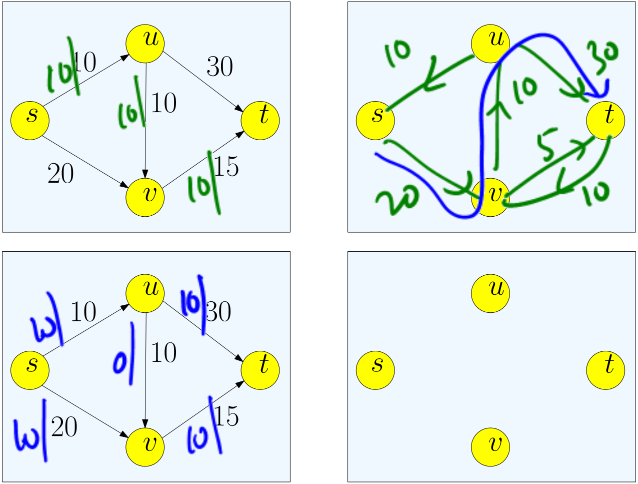

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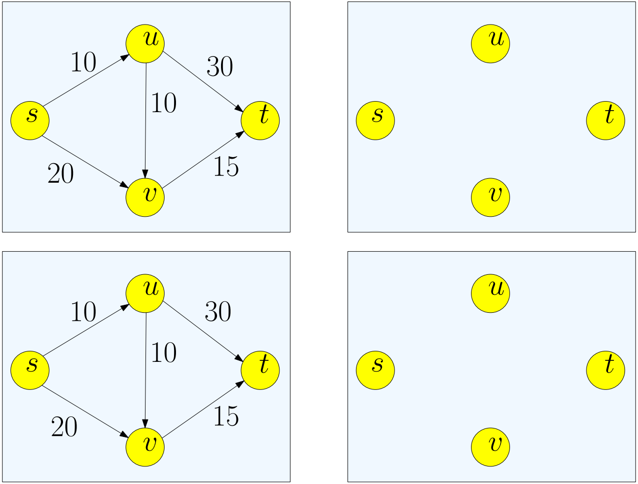

Properties about Augmentation: Flow

Lemma

If f *is a flow and* P *is a simple* s-t *path in* G_f*, then* $f' = \texttt{augment}(f, P)$ *is also a flow.*

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2. Conservation constraint: Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide).

□

Properties of Augmentation

Conservation Constraint

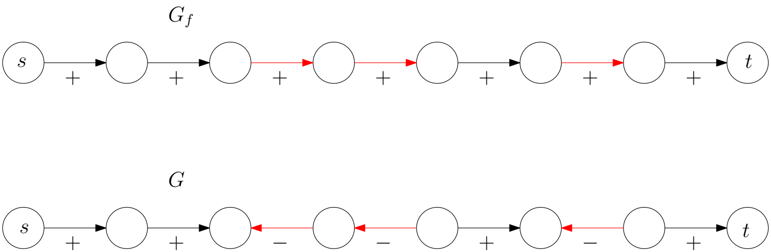

Figure: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

Properties of Augmentation

Integer Flow

Lemma

At every stage of the Ford-Fulkerson algorithm, the flow values on the edges (i.e., $f(e)$, for all edges e) and the residual capacities in G_f are integers.

Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for j iterations. Then in $(j+1)$st iteration, minimum capacity edge b is an integer, and so flow after augmentation is an integer. □

Progress in Ford-Fulkerson

Proposition

Let f be a flow and f' be flow after one augmentation. Then $v(f) < v(f')$.

Proof.

Let P be an augmenting path, i.e., P is a simple s-t path in residual graph. We have the following.

1. First edge e in P must leave s.
2. Original network G has no incoming edges to s; hence e is a forward edge.
3. P is simple and so never returns to s.
4. Thus, value of flow increases by the flow on edge e. □

Termination proof for integral flow

Theorem

Let C be the minimum cut value; in particular $C \leq \sum_{e \text{ out of } s} c(e)$. Ford-Fulkerson algorithm terminates after finding at most C augmenting paths.

Proof.

The value of the flow increases by at least $\mathbf{1}$ after each augmentation. Maximum value of flow is at most C. □

Running time

1. Number of iterations $\leq C$.
2. Number of edges in $G_f \leq 2m$.
3. Time to find augmenting path is $O(n + m)$.
4. Running time is $O(C(n + m))$ (or $O(mC)$).

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?

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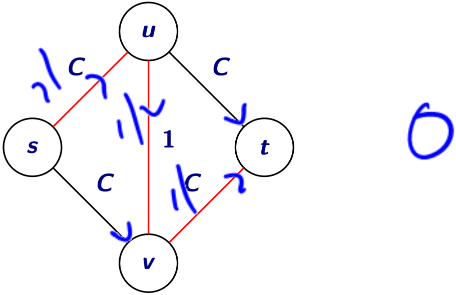

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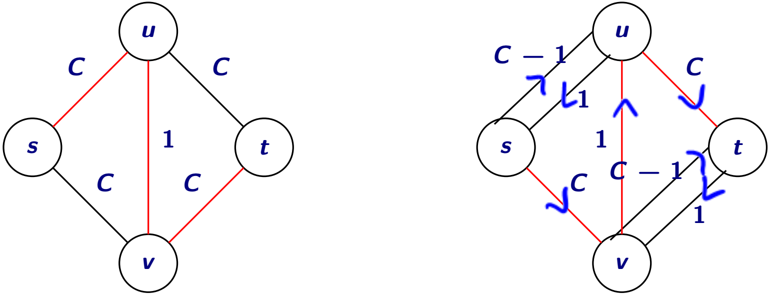

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Correctness of Ford-Fulkerson

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Question: When the algorithm terminates, is the flow computed the maximum s-t flow?

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Why the augmenting path approach works

Question: When the algorithm terminates, is the flow computed the maximum s-t flow?

Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

Recalling Cuts

Definition

Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \to t$ path in $E - E'$. **Capacity** of cut E' is $\sum_{e \in E'} c(e)$.

Let $A \subset V$ such that

1. $s \in A$, $t \notin A$, and
2. $B = V \setminus -A$ and hence $t \in B$.

Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim

(A, B) is an s-t cut.

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

Ford-Fulkerson Correctness

Lemma

If there is no s-t path in G_f then there is some cut (A, B) such that $v(f) = c(A, B)$

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Proof.

Let A be all vertices reachable from s in G_f; $B = V \setminus A$.

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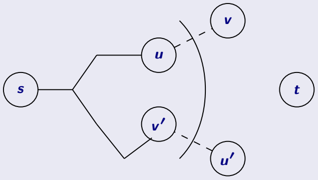

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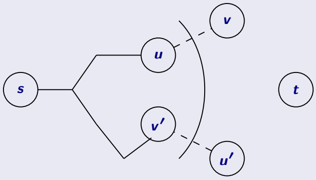
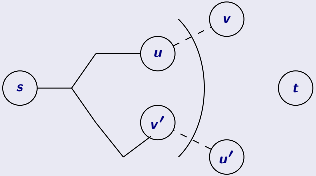


1. $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G.
2. If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (saturated edge) because otherwise v is reachable from s in G_f.

□

Lemma Proof Continued

Proof.

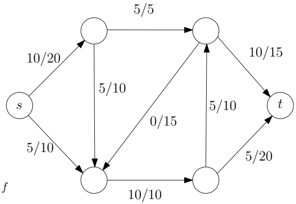


1. If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
2. Thus,

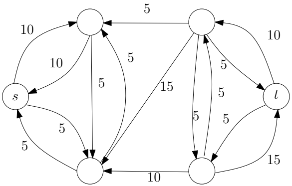
$$
\begin{aligned}
v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\
&= f^{\text{out}}(A) - 0 \\
&= c(A, B) - 0 \\
&= c(A, B).
\end{aligned}
$$

□

Example

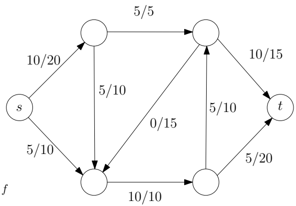


Flow f



Residual graph G_f: no s-t path

Example



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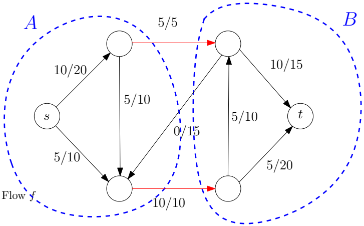

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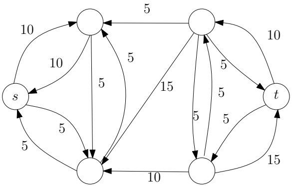

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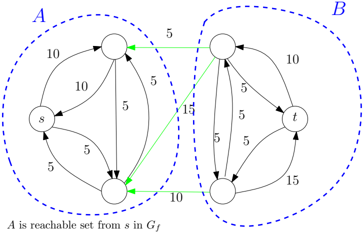

A is reachable set from s in G_f

Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

Proof.

1. For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$.
2. For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-t cut (A^*, B^*).
3. Hence, f^* is maximum.

□

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.

Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G with integer capacities, there is a maximum s-t flow that is integer valued.

Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

Does it terminate?

(A) **algFordFulkerson** always terminates.

(B) **algFordFulkerson** might not terminate if the input has real numbers.

(C) **algFordFulkerson** might not terminate if the input has rational numbers.

(D) **algFordFulkerson** might not terminate if the input is only integer numbers that are sufficiently large.

Finding a Minimum Cut

Question: How do we find an actual minimum s-t cut?

Finding a Minimum Cut

Question: How do we find an actual minimum s-t cut?
Proof gives the algorithm!

1. Compute an s-t maximum flow f in G
2. Obtain the residual graph G_f
3. Find the nodes A reachable from s in G_f
4. Output the cut $(A, B) = \{(u, v) \mid u \in A, v \in B\}$. Note: The cut is found in G while A is found in G_f

Running time is essentially the same as finding a maximum flow.

Note: Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the upper bound be achieved?

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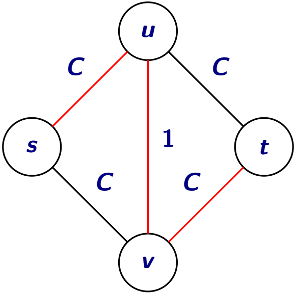

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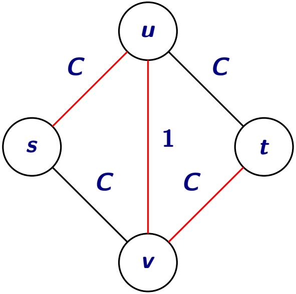

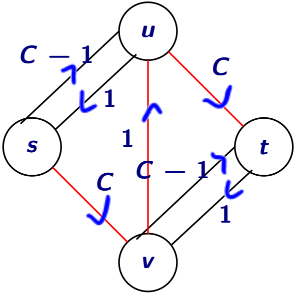

Polynomial Time Algorithms

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Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

1. Choose the augmenting path with largest bottleneck capacity.
2. Choose the shortest augmenting path.

Part II

Polynomial-time Augmenting Path Algorithms

Augmenting along high capacity paths

Definition

Given $G = (V, E)$ with edge capacities and a path P, the bottlneck capacity of P is smallest capacity among edges of P.

Algorithm: In each iteration of Ford-Fulkerson choose an augmenting path with largest bottleneck capacity.

Question: How many iterations does the algorithm take?

Finding path with largest bottleneck capacity

G_f - residual network with (residual) capacities.
n vertices and m edges.
Finding the s-t path with largest bottleneck capacity can be done (faster is better) in:

(A) $O(n + m)$

(B) $O(m + n \log n)$

(C) $O(nm)$

(D) $O(m^2)$

(E) $O(m^3)$

time (expected or deterministic is fine here).

Augmenting Paths with Large Bottleneck Capacity

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   3. Check if there is a path from s to t
   4. Do binary search to find largest Δ
   5. Running time: $O(m \log C)$
   6. Max bottleneck capacity is one of the edge capacities. Why?
   7. Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
   8. Algorithm's running time is $O(m \log m)$.
   9. Alternative algorithm: modify Dijkstra to get $O(m + n \log n)$.

Analyzing number of iterations

$G = (V, E)$ flow network with integer capacities. F^* is max s-t-flow value.

Theorem

Algorithm terminates in $O(m \log F^*)$ *iterations.*

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Suppose algorithm takes k iterations. Let α_i be flow value after i iterations. $\alpha_0 = 0$. In Ford-Fulkerson we have $\alpha_{i+1} \geq \alpha_i + 1$. For the new algorithm we have,

Lemma

If algorithm does not terminate after the i'th iteration, amount of flow augmented in $(i+1)$st iteration is at least $\min\{1, (F^* - \alpha_i)/m\}$. Hence, $\alpha_{i+1} - \alpha_i \geq \min\{1, (F^* - \alpha_i)/m\}$.

Analyzing number of iterations

Assume lemma. Let $\beta_i = F^* - \alpha_i$ be residual flow left after i iterations. We have $\beta_0 = F^*$.

$$\alpha_{i+1} - \alpha_i = \beta_i - \beta_{i+1} \geq \beta_i/m$$

$\leq \frac{F^* - \alpha_i}{m}$; $\geq$

implies

$$\beta_{i+1} \leq (1 - 1/m)\beta_i$$

Therefore, for $k \geq 1$,

$$\beta_k \leq (1 - 1/m)^k \beta_0 \leq (1 - 1/m)^k F^*$$

$$\left(1 - \frac{1}{m}\right)^{m \ln F^*} \leq \left[\left(1 - \frac{1}{m}\right)^m\right]^{\ln F^*} \leq \left(\frac{1}{e}\right)^{\ln F^*} \leq \frac{1}{F^*}$$

Analyzing number of iterations

Assume lemma. Let $\beta_i = F^* - \alpha_i$ be residual flow left after i iterations. We have $\beta_0 = F^*$.

$$\alpha_{i+1} - \alpha_i = \beta_i - \beta_{i+1} \leq \beta_i / m$$

implies

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Therefore, for $k \geq 1$,

$$\beta_k \leq (1 - 1/m)^k \beta_0 \leq (1 - 1/m)^k F^*$$

Thus, after $h = m \ln F^*$ iterations,
$\beta_h \leq (1 - 1/m)^{m \ln F^*} F^* \leq exp(-\ln F^*) F^* \leq 1$.
This implies that algorithm terminates in $1 + m \ln F^*$ iterations.
And $F^* \leq mC$ and hence algorithm terminates in $O(m \log mC)$ iterations.

Proof of Lemma

- f_i flow in G after i iterations of value α_i. G_{f_i} is residual graph.
- In G_{f_i} there is a flow of value $F^* - \alpha_i$.
- Do a flow decomposition in G_{f_i} on at most m paths.
- Implies that there is a flow of value $F^* - \alpha_i$ in G_{f_i} that can be decomposed into at most m paths.
- One of those paths, say P, carries at least $(F^* - \alpha_i)/m$ flow
- Flow on max bottleneck path must be at least as large as that on P. This implies that the amount of augmentation that the algorithm does in iteration $i + 1$ is at least $(F^* - \alpha_i)/m$.
- Thus, $\alpha_{i+1} \geq \alpha_i + (F^* - \alpha_i)/m$.

Running time analysis

- Each iteration requires finding a max bottleneck capacity path in residual graph. Can be found in $O(n \log n + m)$ or in $O(m \log C)$ time.
- Number of iterations is $O(m \log mC)$.
- Hence overall running time is $O(m^2 \log mC \log C)$ or $O(mn \log n \log mC + m^2 \log mC)$.

Strongly polynomial time algorithm

Many problems has inputs with two types of information:

- combinatorial
- numerical

Example:
Graph problems: vertices and edges are combinatorial part and edge/vertex lengths/capacities are numerical.

An algorithm for a problem is called *strongly polynomial* if its running time is a polynomial and it does not depend on the numerical part. Here, we assume that standard arithmetic operations on the input numbers takes constant time. Otherwise it is *weakly polynomial*. It is *pseudo-polynomial* if the run-time is polynomial assuming numerical data is in unary.

A strongly polynomial time algorithm for max flow

Algorithm: In each iteration of Ford-Fulkerson choose a shortest augmenting path in the residual graph.

```
algEdmondsKarp
    for every edge e, f(e) = 0
    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
        Perform BFS in G_f
        P: shortest s-t path in G_f
        f = augment(f, P)
        Construct new residual graph G_f.
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```

Theorem

Algorithm terminates in $O(mn)$ iterations. Thus, overall running time is $O(m^2 n)$.

Orlin's Algorithm

- Currently, fastest strongly polynomial time algorithm runs in $O(mn)$ time.
- $O(mn)$ time is also sufficient to do flow-decomposition

You can state and use the above results in a black box fashion when using maximum flow algorithms in reductions.