Network Flows

Lecture 13

October 7, 2016

How many edges to cut?

For the graph depicted on the right. How many edges have to be cut before there is no path between ***s*** and ***t***:

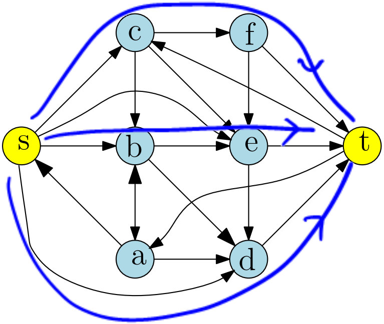
(A) 1

(B) 2

(C) 3

(D) 4

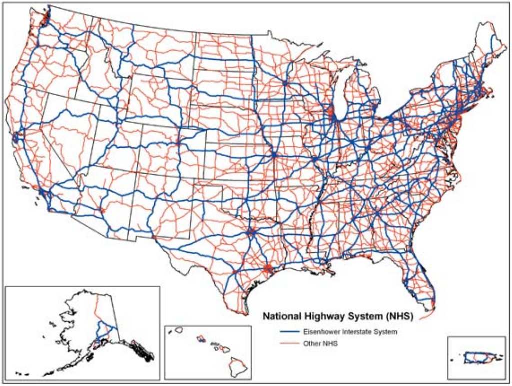
(E) 5



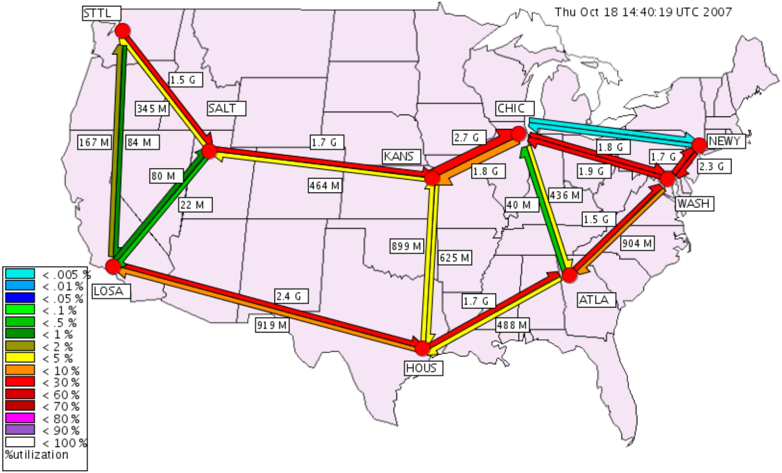
Part I

Network Flows: Introduction and Setup

Transportation/Road Network



Internet Backbone Network



Common Features of Flow Networks

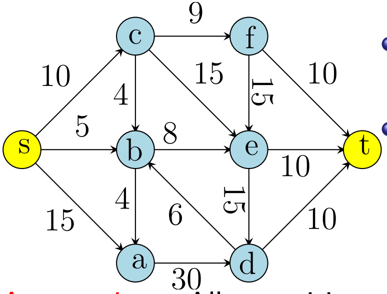
1. **Network** represented by a (directed) *graph* $G = (V, E)$.
2. Each edge e has a **capacity** $c(e) \geq 0$ that limits amount of *traffic* on e.
3. *Source(s)* of traffic/data.
4. *Sink(s)* of traffic/data.
5. Traffic *flows* from sources to sinks.
6. Traffic is *switched/interchanged* at nodes.

Flow abstract term to indicate stuff (traffic/data/etc) that **flows** from sources to sinks.

Single Source/Single Sink Flows

Simple setting:

- Single source s and single sink t.
- Every other node v is an **internal** node.
- Flow originates at s and terminates at t.



- Each edge e has a capacity $c(e) \geq 0$.
- Sometimes assume: Source $s \in V$ has no incoming edges, and sink $t \in V$ has no outgoing edges.

Assumptions: All capacities are integer, and every vertex has at least one edge incident to it.

Definition of Flow

Two ways to define flows:

1. edge based, or
2. path based.

Essentially equivalent but have different uses.

Edge based definition is more compact.

Edge Based Definition of Flow

Definition

Flow in network $G = (V, E)$, is function $f : E \to \mathbb{R}^{\geq 0}$ s.t.

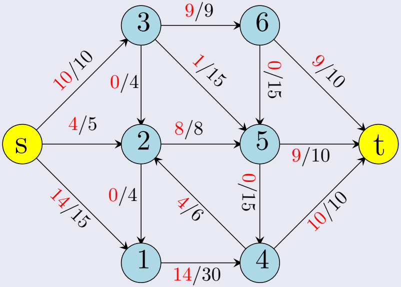


Figure: Flow with value.

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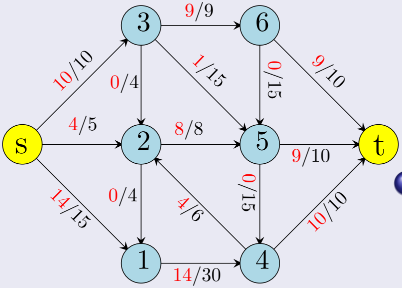

Figure: Flow with value.

1. Capacity Constraint: For each edge e, $f(e) \leq c(e)$.

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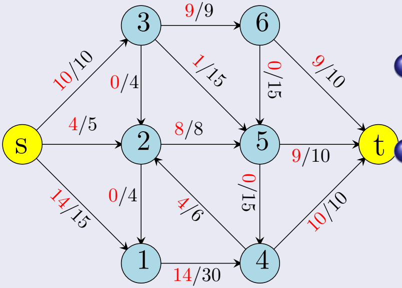

1. Capacity Constraint: For each edge e, $f(e) \leq c(e)$.
2. Conservation Constraint: For each vertex $v \neq s, t$.

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

Figure: Flow with value.

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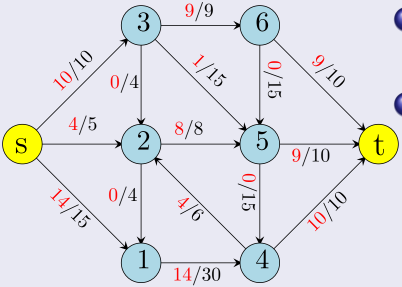

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3. Value of flow= (total flow out of source) − (total flow in to source).

Flow...

Conservation of flow law is also known as **Kirchhoff's law**.

More Definitions and Notation

Notation

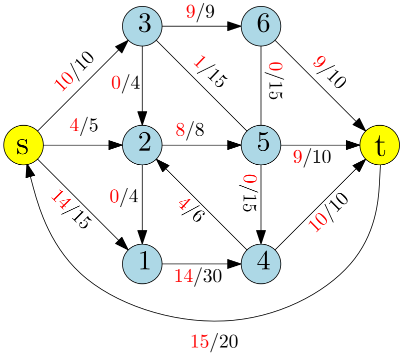
1. The inflow into a vertex v is $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$ and the outflow is $f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$
2. For a set of vertices A, $f^{\text{in}}(A) = \sum_{e \text{ into } A} f(e)$. Outflow $f^{\text{out}}(A)$ is defined analogously

Definition

For a network $G = (V, E)$ with source s, the **value** of flow f is defined as $v(f) = f^{\text{out}}(s) - f^{\text{in}}(s)$.

In the flow depicted on the right, the value of the flow is.

- **(A)** 6.
- **(B)** 13.
- **(C)** 18.
- **(D)** 28.
- **(E)** 43.



A Path Based Definition of Flow

Intuition: Flow goes from source s to sink t along a path.

$\mathcal{P}$: set of all paths from s to t. $|\mathcal{P}|$ can be **exponential** in n.

Definition (Flow by paths.)

A **flow** in network $G = (V, E)$, is function $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ s.t.

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$$\sum_{p \in \mathcal{P} : e \in p} f(p) \leq c(e)$$

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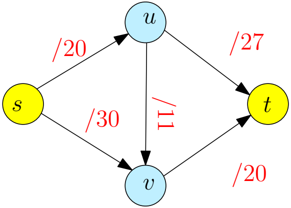
Value of flow: $\sum_{p \in \mathcal{P}} f(p)$.

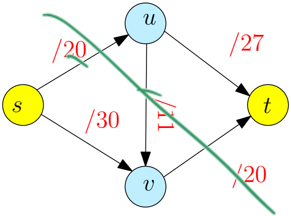
Example



$\mathcal{P} = \{p_1, p_2, p_3\}$
$p_1 : s \to u \to t$
$p_2 : s \to u \to v \to t$
$p_3 : s \to v \to t$

$f(p_1) = 10, f(p_2) = 4, f(p_3) = 6$

Example



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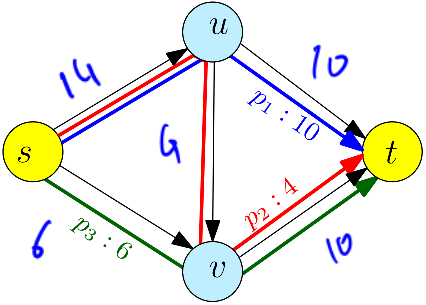

Path based flow implies edge based flow

Lemma

Given a path based flow $f : \mathcal{P} \to \mathbb{R}^{\geq 0}$ *there is an edge based flow* $f' : E \to \mathbb{R}^{\geq 0}$ *of the same value.*

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Lemma

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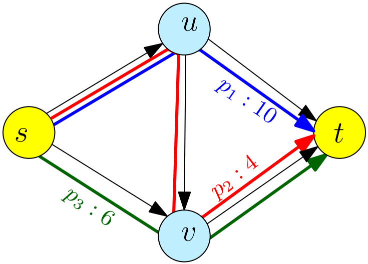
Proof.

For each edge e define $f'(e) = \sum_{p: e \in p} f(p)$.

Exercise: Verify capacity and conservation constraints for f'.

Exercise: Verify that value of f and f' are equal □

Example



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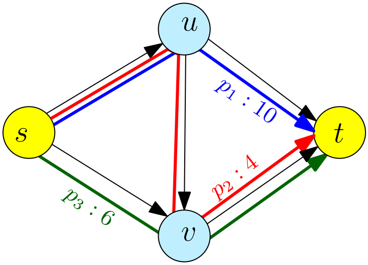

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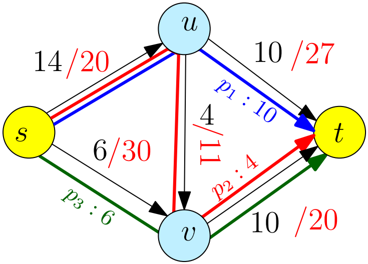

$f'(s \to u) = 14$
$f'(u \to v) = 4$
$f'(s \to v) = 6$
$f'(u \to t) = 10$
$f'(v \to t) = 10$

Flow Decomposition

Edge based flow to Path based Flow

Lemma

Given an edge based flow $f' : E \to \mathbb{R}^{\geq 0}$*, there is a path based flow* $f : \mathcal{P} \to \mathbb{R}^{\geq 0}$ *of same value. Moreover,* f *assigns non-negative flow to at most* m *paths where* $|E| = m$ *and* $|V| = n$*. Given* f'*, the path based flow can be computed in* $O(mn)$ *time.*

Flow Decomposition

Edge based flow to Path based Flow

Proof Idea.

1. Remove all edges with $f'(e) = 0$.
2. Find a path p from s to t.
3. Assign $f(p)$ to be $\min_{e \in p} f'(e)$.
4. Reduce $f'(e)$ for all $e \in p$ by $f(p)$.
5. Repeat until no path from s to t.
6. In each iteration at least on edge has flow reduced to zero.

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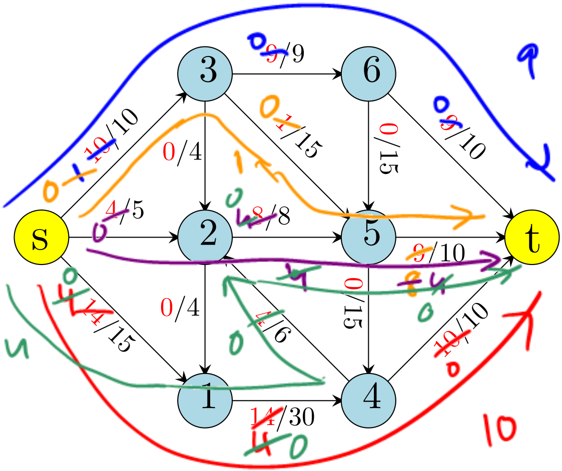
4. Reduce $f'(e)$ for all $e \in p$ by $f(p)$.
5. Repeat until no path from s to t.
6. In each iteration at least on edge has flow reduced to zero.
7. Hence, at most m iterations. Can be implemented in $O(m(m + n))$ time. $O(mn)$ time requires care. □

Example


s
t
1
2
3
4
5
6
10/10
4/5
14/15
9/9
0/4
1/15
8/8
0/15
9/10
0/4
4/6
0/15
9/10
10/10
14/30

Edge vs Path based Definitions of Flow

Edge based flows:

1. **compact** representation, only m values to be specified, and
2. need to check flow conservation explicitly at each internal node.

Path flows:

1. in some applications, paths more natural,
2. not compact,
3. no need to check flow conservation constraints.

Equivalence shows that we can go back and forth easily.

The Maximum-Flow Problem

Problem

Input A network G with capacity c and source s and sink t.

Goal Find flow of **maximum** value.

The Maximum-Flow Problem

Problem

Input: A network G with capacity c and source s and sink t.

Goal: Find flow of **maximum** value.

Question: Given a flow network, what is an *upper bound* on the maximum flow between source and sink?

Cuts

Definition (s-t cut)

Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \to t$ path in $E - E'$.
The **capacity** of a cut E' is $c(E') = \sum_{e \in E'} c(e)$.

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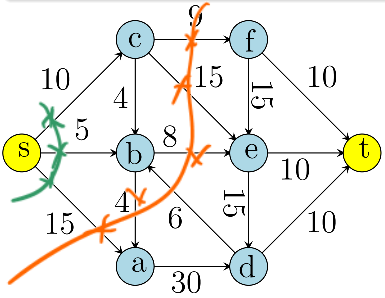

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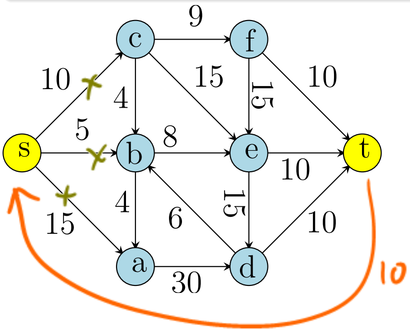
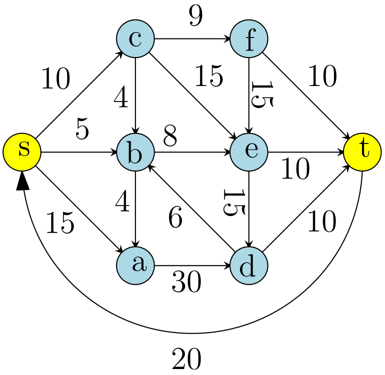


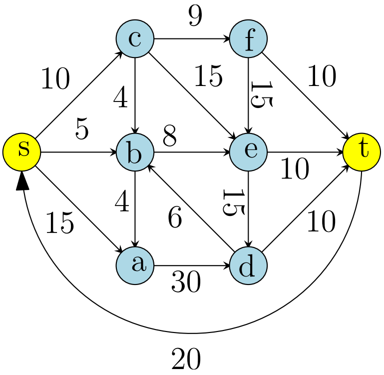
Caution:

1. Cut may leave $t \to s$ paths!
2. There might be many s-t cuts.

s — t cuts

A death by a thousand cuts

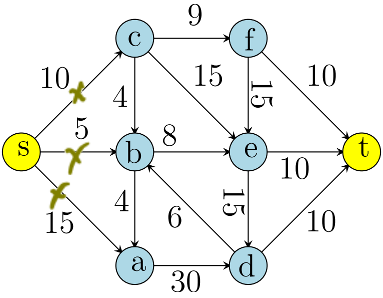




Minimal Cut

Definition (Minimal s-t cut.)

Given a s-t flow network $G = (V, E)$, $E' \subseteq E$ is a **minimal cut** if for all $e \in E'$, if $E' \setminus \{e\}$ is not a cut.



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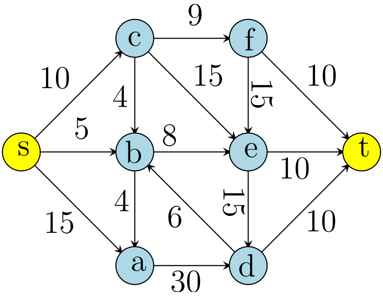


Observation: given a cut E', can check efficiently whether E' is a minimal cut or not. How?

Is this a minimal cut?

Definition (Minimal s-t cut.)

Given a s-t flow network $G = (V, E)$ with n vertices and m edges, $E' \subseteq E$ is a **minimal cut** if for all $e \in E'$, $E' \setminus \{e\}$ is not a cut.

Checking if a set E' forms a minimal s-t cut can be done in

(A) $O(n + m)$.

(B) $O(n \log n + m)$.

(C) $O((n + m) \log n)$.

(D) $O(nm)$.

(E) $O(nm \log n)$.

(F) You flow, me cut.

Cuts as Vertex Partitions

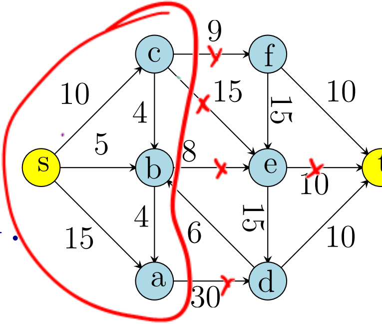
Let $A \subset V$ such that

1. $s \in A$, $t \notin A$, and
2. $B = V \setminus A$ (hence $t \in B$).

The **cut** (A, B) is the set of edges

$$(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}.$$

Cut (A, B) is set of edges leaving A.



Claim

(A, B) *is an* s-t *cut.*

Proof.

Let P be any $s \to t$ path in G. Since t is not in A, P has to leave A via some edge (u, v) in (A, B). □

Cuts as Vertex Partitions

Lemma

Suppose E' *is an* s-t *cut. Then there is a cut* (A, B) *such that* $(A, B) \subseteq E'$.

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Suppose E' is an s-t cut. Then there is a cut (A, B) such that $(A, B) \subseteq E'$.

Proof.

E' is an s-t cut implies no path from s to t in $(V, E - E')$.

1. Let A be set of all nodes reachable by s in $(V, E - E')$.
2. Since E' is a cut, $t \not\in A$.
3. $(A, B) \subseteq E'$. Why?

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Corollary

Every minimal *s-t cut E' is a cut of the form (A, B).*

Alternate notation for cuts

Other common notation for cuts:

Undirected graphs: $G = (V, E)$ and $A \subset V$. $\delta_G(A)$ or $\delta(A)$ is set of edges with one end point in A and the other end point in $V \setminus A$.

Directed graphs: $G = (V, E)$ and $A \subset V$.

$$\delta_G^+(A) = \{(u, v) \in E \mid u \in A, v \in V \setminus A\}$$

and

$$\delta_G^-(A) = \{(u, v) \in E \mid u \in V \setminus A, v \in A\}$$

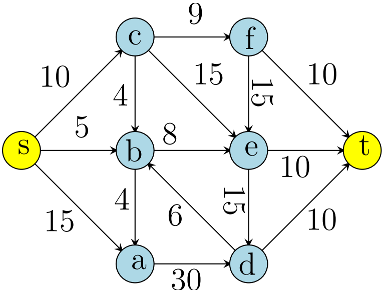
Minimum Cut

Definition

Given a flow network an s-t **minimum** cut is a cut E' of smallest capacity amongst all s-t cuts.

The minimum cut in the network flow depicted is:

- **(A)** 10
- **(B)** 18
- **(C)** 28
- **(D)** 30
- **(E)** 48.
- **(F)** No minimum cut, no cry.



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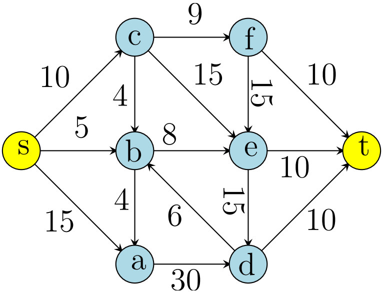

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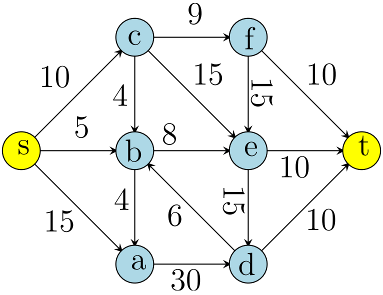

Observation: exponential number of ***s*-*t*** cuts and no "easy" algorithm to find a minimum cut.

The Minimum-Cut Problem

Problem

Input: A flow network G

Goal: Find the capacity of a *minimum* s-t cut

Flows and Cuts

Lemma

For any s-t cut E', **maximum** *s-t flow $\leq$ capacity of E'.*

Proof.

Formal proof easier with path based definition of flow.
Suppose $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ is a max-flow.

Flows and Cuts

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Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why?

Flows and Cuts

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Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why?
Assign each path $p \in \mathcal{P}$ to exactly one edge $e \in E'$.
Let $\mathcal{P}_e$ be paths assigned to $e \in E'$.

Flows and Cuts

Lemma

For any s-t *cut* E', **maximum** s-t *flow* $\leq$ *capacity of* E'.

Proof.

Formal proof easier with path based definition of flow.
Suppose $f : \mathcal{P} \rightarrow \mathbb{R}^{\geq 0}$ is a max-flow.

Every path $p \in \mathcal{P}$ contains an edge $e \in E'$. Why?
Assign each path $p \in \mathcal{P}$ to exactly one edge $e \in E'$.
Let $\mathcal{P}_e$ be paths assigned to $e \in E'$. Then

$$v(f) = \sum_{p \in \mathcal{P}} f(p) = \sum_{e \in E'} \sum_{p \in \mathcal{P}_e} f(p) \leq \sum_{e \in E'} c(e).$$

□

Flows and Cuts

Lemma

For any s-t *cut* E', **maximum** s-t *flow* $\leq$ *capacity of* E'.

Corollary

Maximum s-t *flow* $\leq$ *minimum* s-t *cut.*

Max-Flow Min-Cut Theorem

Theorem

In any flow network the maximum s-t *flow is equal to the minimum* s-t *cut.*

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Can compute minimum-cut from maximum flow and vice-versa!

Max-Flow Min-Cut Theorem

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In any flow network the maximum s-t flow is equal to the minimum s-t cut.

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Proof coming shortly.
Many applications:

1. optimization
2. graph theory
3. combinatorics

The Maximum-Flow Problem

Problem

Input A network G with capacity c and source s and sink t.

Goal Find flow of **maximum** value from s to t.

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Exercise: Given G, s, t as above, show that one can remove all edges into s and all edges out of t without affecting the flow value between s and t.