Part I

Hash Tables
Dictionary Data Structure

1. $\mathcal{U}$: universe of keys with total order: numbers, strings, etc.
2. Data structure to store a subset $S \subseteq \mathcal{U}$
3. **Operations:**
   1. **Search/look up:** given $x \in \mathcal{U}$ is $x \in S$?
   2. **Insert:** given $x \not\in S$ add $x$ to $S$.
   3. **Delete:** given $x \in S$ delete $x$ from $S$
4. **Static** structure: $S$ given in advance or changes very infrequently, main operations are lookups.
5. **Dynamic** structure: $S$ changes rapidly so inserts and deletes as important as lookups.

Can we do everything in $O(1)$ time?
Hashing and Hash Tables

Hash Table data structure:
1. A (hash) table/array $T$ of size $m$ (the table size).
2. A hash function $h : \mathcal{U} \to \{0, \ldots, m - 1\}$.
3. Item $x \in \mathcal{U}$ hashes to slot $h(x)$ in $T$. 

Given $S \subseteq \mathcal{U}$. How do we store $S$ and how do we do lookups?

Ideal situation:
1. Each element $x \in S$ hashes to a distinct slot in $T$.
2. Lookup: Given $y \in \mathcal{U}$ check if $T[h(y)] = y$.

$O(1)$ time!

Collisions unavoidable if $|T| < |\mathcal{U}|$. Several techniques to handle them.
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Collisions unavoidable if $|T| < |\mathcal{U}|$. Several techniques to handle them.
Handling Collisions: Chaining

Collision: \( h(x) = h(y) \) for some \( x \neq y \).

Chaining/Open hashing to handle collisions:

1. For each slot \( i \) store all items hashed to slot \( i \) in a linked list. \( T[i] \) points to the linked list

2. Lookup: to find if \( y \in U \) is in \( T \), check the linked list at \( T[h(y)] \). Time proportion to size of linked list.

Does hashing give \( O(1) \) time per operation for dictionaries?
Hash Functions

Parameters: \( N = |\mathcal{U}| \) (very large), \( m = |T| \), \( n = |S| \)
Goal: \( O(1) \)-time lookup, insertion, deletion.

**Single hash function**

If \( N \geq m^2 \), then for any hash function \( h : \mathcal{U} \rightarrow T \) there exists \( i < m \) such that at least \( N/m \geq m \) elements of \( \mathcal{U} \) get hashed to slot \( i \).
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Such a bad set may lead to \( O(m) \) lookup time!
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Lesson:

- Consider a family \( \mathcal{H} \) of hash functions with good properties and choose \( h \) uniformly at random.
- Guarantees: small \# collisions in expectation for a given \( S \).
- \( \mathcal{H} \) should allow efficient sampling.
Question: What are good properties of $H$ in distributing data?

1. Uniform: Consider any element $x \in U$. Then if $h \in H$ is picked randomly then $x$ should go into a random slot in $T$. In other words $\Pr[h(x) = i] = \frac{1}{m}$ for every $0 \leq i < m$.

2. Universal: Consider any two distinct elements $x, y \in U$. Then if $h \in H$ is picked randomly then the probability of a collision between $x$ and $y$ should be at most $\frac{1}{m}$. In other words $\Pr[h(x) = h(y)] = \frac{1}{m}$ (cannot be smaller).

3. Second property is stronger than the first and the crucial issue.

Definition: A family of hash functions $H$ is ($2$-)universal if for all distinct $x, y \in U$, $\Pr[h \sim H][h(x) = h(y)] = \frac{1}{m}$ where $m$ is the table size.
Question: What are good properties of $\mathcal{H}$ in distributing data?

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Universal Hashing

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Definition

A family of hash function $\mathcal{H}$ is (2-)universal if for all distinct $x, y \in \mathcal{U}$, $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] = 1/m$ where $m$ is the table size.
Analyzing Universal Hashing

Question: Fixing set $S$, what is the expected time to look up $x \in S$ when $h$ is picked uniformly at random from $\mathcal{H}$?

1. $\ell(x)$: the size of the list at $T[h(x)]$. We want $E[\ell(x)]$

2. For $y \in S$ let $D_y$ be one if $h(y) = h(x)$, else zero.
   
   $\ell(x) = \sum_{y \in S} D_y$

   
   $E[\ell(x)] = \sum_{y \in S} E[D_y] = \sum_{y \in S} \Pr[h(x) = h(y)] = \sum_{y \in S} \frac{1}{m}$ (since $\mathcal{H}$ is a universal hash family)

   $= \frac{|S|}{m} \leq 1$ if $|S| \leq m$
Analyzing Universal Hashing

**Question:** Fixing set $S$, what is the *expected* time to look up $x \in S$ when $h$ is picked uniformly at random from $\mathcal{H}$?

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$$\ell(x) = \sum_{y \in S} D_y$$

$$E[\ell(x)] = \sum_{y \in S} E[D_y] = \sum_{y \in S} Pr[h(x) = h(y)]$$

$$= \sum_{y \in S} \frac{1}{m} \quad \text{(since $\mathcal{H}$ is a universal hash family)}$$

$$= \frac{|S|}{m} \leq 1 \quad \text{if } |S| \leq m$$
Question: What is the expected time to look up $x$ in $T$ using $h$ assuming chaining used to resolve collisions?

Answer: $O(n/m)$.
Analyzing Universal Hashing

**Question:** What is the *expected* time to look up \( x \) in \( T \) using \( h \) assuming chaining used to resolve collisions?

**Answer:** \( O(n/m) \).

**Comments:**

1. \( O(1) \) expected time also holds for insertion.
2. Analysis assumes static set \( S \) but holds as long as \( S \) is a set formed with at most \( O(m) \) insertions and deletions.
3. **Worst-case:** look up time can be large! How large? \( \Omega(\log n/\log \log n) \)
Universal Hash Family

Universal: \( \mathcal{H} \) such that \( \Pr[h(x) = h(y)] = 1/m \).

All functions

\( \mathcal{H} \): Set of all possible functions \( h : U \rightarrow \{0, \ldots, m - 1\} \).
- Universal.
**Universal Hash Family**

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### All functions

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- Universal.
- \( |\mathcal{H}| = m^{\left\lvert \mathcal{U} \right\rvert} \)
- representing \( h \) requires \( |\mathcal{U}| \log m \) – Not \( O(1) \)!

We need compactly representable universal family.
Universal Hash Family

Universal: $\mathcal{H}$ such that $\Pr[h(x) = h(y)] = 1/m$.

**All functions**

$\mathcal{H}$: Set of all possible functions $h : \mathcal{U} \rightarrow \{0, \ldots, m - 1\}$.
- Universal.
- $|\mathcal{H}| = m^{|\mathcal{U}|}$
- representing $h$ requires $|\mathcal{U}| \log m$ – Not $O(1)$!

We need *compactly representable* universal family.
Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |\mathcal{T}|$, $n = |\mathcal{S}|$

1. Choose a **prime** number $p \geq N$. $\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$ is a field.

2. For $a, b \in \mathbb{Z}_p$, $a \neq 0$, define the hash function $h_{a,b}$ as

   $$h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m.$$

3. Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$. Note that $|\mathcal{H}| = p(p - 1)$. 

Theorem $\mathcal{H}$ is a universal hash family.

Comments:
1. Hash family is of small size, easy to sample from.
2. Easy to store a hash function ($a$, $b$ have to be stored) and evaluate it.
Compact Universal Hash Family

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2. Easy to store a hash function (\( a, b \) have to be stored) and evaluate it.
Some math required...

Lemma (LemmaUnique)

Let $p$ be a prime number, $x$: an integer number in $\{1, \ldots, p-1\}$.
$\implies$ There exists a unique $y$ s.t. $xy = 1 \mod p$.

In other words: For every element there is a unique inverse.
$\implies$ $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ when working modulo $p$ is a field.
Proof of Lemma Unique

Claim

Let $p$ be a prime number. For any $x, y, z \in \{1, \ldots, p - 1\}$ s.t. $y \neq z$, we have that $xy \mod p \neq xz \mod p$.

Proof.

Assume for the sake of contradiction $xy \mod p = xz \mod p$. Then

$$x(y - z) = 0 \mod p$$

$$\implies \quad p \text{ divides } x(y - z)$$

$$\implies \quad p \text{ divides } y - z$$

$$\implies \quad y - z = 0$$

$$\implies \quad y = z.$$
Proof of LemmaUnique

Lemma (LemmaUnique)

Let \( p \) be a prime number, \( x \) an integer number in \( \{1, \ldots, p - 1\} \).

\[ \implies \text{There exists a unique } y \text{ s.t. } xy = 1 \mod p. \]

Proof.

By the above claim if \( xy = 1 \mod p \) and \( xz = 1 \mod p \) then \( y = z \). Hence uniqueness follows.
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Existence. For any \( x \in \{1, \ldots, p - 1\} \) we have that
\[ \{ x \cdot 1 \mod p, x \cdot 2 \mod p, \ldots, x \cdot (p - 1) \mod p \} = \{1, 2, \ldots, p - 1\}. \]
\[ \implies \text{There exists a number } y \in \{1, \ldots, p - 1\} \text{ such that } xy = 1 \mod p. \]
Proof of the Theorem: Outline

\[ h_{a,b}(x) = ((ax + b) \mod p) \mod m. \]

**Theorem**

\[ \mathcal{H} = \{ h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0 \} \text{ is universal.} \]

**Proof.**

Fix \( x, y \in \mathcal{U}. \) We need to show that

\[ \Pr_{h_{a,b} \sim \mathcal{H}}[h_{a,b}(x) = h_{a,b}(y)] \leq 1/m. \]

Note that \(|\mathcal{H}| = p(p - 1)|.\]
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Note that \( |\mathcal{H}| = p(p - 1) \).

1. Let \((a, b)\) (equivalently \( h_{a,b} \)) be bad for \( x, y \) if \( h_{a,b}(x) = h_{a,b}(y) \).

2. **Claim:** Number of bad \((a, b)\) is at most \( p(p - 1)/m \).

3. Total number of hash functions is \( p(p - 1) \) and hence probability of a collision is \( \leq 1/m \).
$g_{a,b}(x) = (ax + b) \mod p$, \hspace{1em} $h_{a,b}(x) = (g_{a,b}(x)) \mod m$

First map $x \neq y$ to $r = g_{a,b}(x)$ and $s = g_{a,b}(y)$. $r \neq s$ (LemmaUnique)
Intuition for the Claim

\[ g_{a,b}(x) = (ax + b) \mod p, \quad h_{a,b}(x) = (g_{a,b}(x)) \mod m \]

First map \( x \neq y \) to \( r = g_{a,b}(x) \) and \( s = g_{a,b}(y) \). \( r \neq s \) (LemmaUnique)

As \((a, b)\) varies, \((r, s)\) takes all possible \( p(p - 1) \) values. Since \((a, b)\) is picked u.a.r., every value of \((r, s)\) has equal probability.
Intuition for the Claim

\[ g_{a,b}(x) = (ax + b) \mod p, \quad h_{a,b}(x) = (g_{a,b}(x)) \mod m \]
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\[ g_{a,b}(x) = (ax + b) \mod p, \quad h_{a,b}(x) = (g_{a,b}(x)) \mod m \]
$g_{a,b}(x) = (ax + b) \mod p$, $h_{a,b}(x) = (g_{a,b}(x)) \mod m$

1. First part of mapping maps $(x, y)$ to a random location $(g_{a,b}(x), g_{a,b}(y))$ in the “matrix”.

2. $(g_{a,b}(x), g_{a,b}(y))$ is not on main diagonal.

3. All blue locations are “bad” – map by $\mod m$ to a location of collusion.

4. But... at most $\frac{1}{m}$ fraction of allowable locations in the matrix are bad.
We need to show at most $1/m$ fraction of bad $h_{a,b}$

\[ h_{a,b}(x) = (((ax + b) \mod p) \mod m) \]

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \mod p$ and $s = (ay + b) \mod p$. 
We need to show at most $1/m$ fraction of bad $h_{a,b}$

$$h_{a,b}(x) = (((ax + b) \mod p) \mod m)$$

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \mod p$ and $s = (ay + b) \mod p$.

- 1-to-1 correspondence between $p(p-1)$ pairs of $(a, b)$ (equivalently $h_{a,b}$) and $p(p-1)$ pairs of $(r, s)$. 

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2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \mod p$ and $s = (ay + b) \mod p$.

1. 1-to-1 correspondence between $p(p-1)$ pairs of $(a, b)$ (equivalently $h_{a,b}$) and $p(p-1)$ pairs of $(r, s)$.

2. Out of all possible $p(p-1)$ pairs of $(r, s)$, at most $p(p-1)/m$ fraction satisfies $r \mod m = s \mod m$. 
Lemma

If \( x \neq y \) then for any \( a, b \in \mathbb{Z}_p \) such that \( a \neq 0 \), we have
\[
ax + b \mod p \neq ay + b \mod p.
\]

Proof.

If \( ax + b \mod p = ay + b \mod p \) then \( a(x - y) \mod p = 0 \) and \( a \neq 0 \) and \( (x - y) \neq 0 \). However, \( a \) and \( (x - y) \) cannot divide \( p \) since \( p \) is prime and \( a < p \) and \( (x - y) < p \).
Lemma

If \( x \neq y \) then for each \((r, s)\) such that \( r \neq s \) and \( 0 \leq r, s \leq p - 1 \) there is exactly one \( a, b \) such that

\[
ax + b \mod p = r \quad \text{and} \quad ay + b \mod p = s.
\]

Proof.

Solve the two equations:

\[
ax + b = r \mod p \quad \text{and} \quad ay + b = s \mod p.
\]

We get \( a = \frac{r - s}{x - y} \mod p \) and \( b = r - ax \mod p \).

One-to-one correspondence between \((a, b)\) and \((r, s)\)
Understanding the hashing

Once we fix \( a \) and \( b \), and we are given a value \( x \), we compute the hash value of \( x \) in two stages:

1. **Compute**: \( r \leftarrow (ax + b) \mod p \).
2. **Fold**: \( r' \leftarrow r \mod m \)

**Collision...**

Given two distinct values \( x \) and \( y \) they might collide only because of folding.

**Lemma**

\( \# \) not equal pairs \((r, s)\) of \( \mathbb{Z}_p \times \mathbb{Z}_p \) that are folded to the same number is \( p(p - 1)/m \).
Folding numbers

Lemma

The number of pairs \((r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p\) such that \(r \neq s\) and \(r \mod m = s \mod m\) (folded to the same number) is \(p(p - 1)/m\).

Proof.

Consider a pair \((r, s) \in \{0, 1, \ldots, p - 1\}^2\) s.t. \(r \neq s\). Fix \(r\):

1. \(a = r \mod m\).
2. There are \(\lceil p/m \rceil\) values of \(s\) that fold into \(a\). That is \(r \mod m = s \mod m\).
3. One of them is when \(r = s\).
4. \(\Rightarrow\) # of colliding pairs \((\lceil p/m \rceil - 1)p \leq (p - 1)p/m\)
Proof of Claim

# of bad pairs is $\frac{p(p-1)}{m}$

Proof.

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

1. Let $r = ax + b \mod p$ and $s = ay + b \mod p$.

2. Collision if and only if $r \mod m = s \mod m$.

3. (Folding error): Number of pairs $(r, s)$ such that $r \neq s$ and $0 \leq r, s \leq p-1$ and $r \mod m = s \mod m$ is $\frac{p(p-1)}{m}$.

4. From previous lemma there is one-to-one correspondence between $(a, b)$ and $(r, s)$. Hence total number of bad $(a, b)$ pairs is $\frac{p(p-1)}{m}$.
Proof of Claim

# of bad pairs is \( \frac{p(p - 1)}{m} \)

Proof.

Let \( a, b \in \mathbb{Z}_p \) such that \( a \neq 0 \) and \( h_{a,b}(x) = h_{a,b}(y) \).

1. Let \( r = ax + b \mod p \) and \( s = ay + b \mod p \).
2. Collision if and only if \( r \mod m = s \mod m \).
3. (Folding error): Number of pairs \( (r, s) \) such that \( r \neq s \) and \( 0 \leq r, s \leq p - 1 \) and \( r \mod m = s \mod m \) is \( \frac{p(p - 1)}{m} \).
4. From previous lemma there is one-to-one correspondence between \( (a, b) \) and \( (r, s) \). Hence total number of bad \( (a, b) \) pairs is \( \frac{p(p - 1)}{m} \).

Prob of \( x \) and \( y \) to collide:

\[
\frac{\text{# bad } (a, b) \text{ pairs}}{\text{#(a, b) pairs}} = \frac{\frac{p(p-1)}{m}}{p(p-1)} = \frac{1}{m}.
\]
Look up Time

Say $|S| = |T| = m$.
For $0 \leq i \leq m - 1$, $\ell(i)$ : number of elements hashed to slot $i$ in $T$.

Expected look up time

Since for $x \neq y$, $\Pr[h_{a,b}(x) = h_{a,b}(y)] = 1/m$, we get $E[\ell(i)] = |S|/m = 1$. 
Look up Time

Say $|S| = |T| = m$.
For $0 \leq i \leq m - 1$, $\ell(i)$: number of elements hashed to slot $i$ in $T$.

Expected look up time

Since for $x \neq y$, $\Pr[h_{a,b}(x) = h_{a,b}(y)] = 1/m$, we get $E[\ell(i)] = |S|/m = 1$.

Expected worst case look up time

Like in Balls & Bins, $E\left[\max_{i=0}^{m-1} \ell(i)\right] \geq O(\ln n / \ln \ln n)$. 
Look up Time

Say \( |S| = |T| = m \).
For \( 0 \leq i \leq m - 1 \), \( \ell(i) \): number of elements hashed to slot \( i \) in \( T \).

**Expected look up time**

Since for \( x \neq y \), \( \Pr[h_{a,b}(x) = h_{a,b}(y)] = 1/m \), we get
\[
E[\ell(i)] = \frac{|S|}{m} = 1.
\]

**Expected worst case look up time**

Like in *Balls & Bins*,
\[
E \left[ \max_{i=0}^{m-1} \ell(i) \right] \geq O(\ln n / \ln \ln n).
\]

**What if \( |T| = m^2 \) (# Bins is \( m^2 \))**

**Claim:** If \( |T| = m^2 \), then
\[
E \left[ \max_{i=0}^{m-1} \ell(i) \right] = O(1).
\]
Perfect Hashing

Two levels of hash tables

**Question:** Can we make look up time $O(1)$ in worst case?

**Perfect Hashing for Static Data**

- Do hashing once.
- If $Y_i = |\ell(i)| > 10$ then hash elements of $\ell(i)$ to a table of $Y_i^2$ size.
Perfect Hashing
Two levels of hash tables

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Perfect Hashing for Static Data

- Do hashing once.
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Lemma

Worst case expected lookup time is $O(1)$. 
Perfect Hashing
Two levels of hash tables

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**Perfect Hashing for Static Data**
- Do hashing once.
- If \(Y_i = |\ell(i)| > 10\) then hash elements of \(\ell(i)\) to a table of \(Y_i^2\) size.

**Lemma**
*Worst case expected look up time is \(O(1)\).*

**Lemma**
*If \(|S| = O(m)\) then space usage of perfect hashing is \(O(m)\).*
Intuition: Throwing \( m \) Balls in to \( m^2 \) Bins

- \( \Pr[\text{ith ball lands in jth bin}] \)
Intuition: Throwing $m$ Balls in to $m^2$ Bins

- $\Pr[\text{ith ball lands in jth bin}] = 1/m^2$
- For a fixed bin $j$, $Y_j = \# \text{ balls in bin } j$. 

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Intuition: Throwing $m$ Balls in to $m^2$ Bins

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Intuition: Throwing \( m \) Balls in to \( m^2 \) Bins

- \( \Pr[\text{ith ball lands in jth bin}] = 1/m^2 \)
- For a fixed bin \( j \), \( Y_j = \# \text{ balls in bin } j \). \( \mathbb{E}[Y_j] = 1/m \).
- For \( c \geq 3 \), let \( \delta = cm - 1 \). \( \Pr[Y_j > c] \)

\[
\Pr[Y_j > cm/m] = \Pr[Y_j > (1 + \delta) \mathbb{E}[Y_j]] \\
(\text{Chernoff}) < \left( \frac{e^\delta}{(1+\delta)^{(1+\delta)}} \right)^{\mu} \\
= \left( \frac{e^{(cm-1)}}{(cm)^{cm}} \right)^{1/m} \leq \left( \frac{e}{c} \right)^c \left( \frac{1}{m^c} \right) \\
\leq 1/m^3
\]
Intuition: Throwing \( m \) Balls into \( m^2 \) Bins

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\leq \quad 1/m^3
\]

- \( \Pr[\max_{j=1}^{m^2} Y_j > c] \leq 1/m \) (Union bound).
- \( \Pr[\max_{j=1}^{m^2} Y_j \leq c] \geq 1 - 1/m \) – (w.h.p.)
- \( E[\max_j Y_j] \leq c + 1 = O(1) \).
Rehashing, amortization and...  
... making the hash table dynamic

So far we assumed fixed \( S \) of size \( \sim m \).

**Question:** What happens as items are inserted and deleted?

1. If \( |S| \) grows to more than \( cm \) for some constant \( c \) then hash table performance clearly degrades.

2. If \( |S| \) stays around \( \sim m \) but incurs many insertions and deletions then the initial random hash function is no longer random enough!

**Solution:**

1. Choose a new table size based on current number of elements in table.
2. Choose a new random hash function and rehash the elements.
3. Discard old table and hash function.

**Question:** When to rebuild? How expensive?
Rehashing, amortization and...
... making the hash table dynamic

So far we assumed fixed $S$ of size $\approx m$.

**Question:** What happens as items are inserted and deleted?

1. If $|S|$ grows to more than $cm$ for some constant $c$ then hash table performance clearly degrades.

2. If $|S|$ stays around $\approx m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

**Solution:** Rebuild hash table periodically!

1. Choose a new table size based on current number of elements in table.

2. Choose a new random hash function and rehash the elements.

3. Discard old table and hash function.

**Question:** When to rebuild? How expensive?
Rebuilding the hash table

1. Start with table size $m$ where $m$ is some estimate of $|S|$ (can be some large constant).

2. If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.

3. If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant $c$ (say 10), rebuild.

The amortize cost of rebuilding to previously performed operations. Rebuilding ensures $O(1)$ expected analysis holds even when $S$ changes. Hence $O(1)$ expected look up/insert/delete time dynamic data dictionary data structure!
Bloom Filters

Hashing:
1. To insert $x$ in dictionary store $x$ in table in location $h(x)$
2. To lookup $y$ in dictionary check contents of location $h(y)$
Bloom Filters

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Bloom Filter: tradeoff space for false positives
1. Storing items in dictionary expensive in terms of memory, especially if items are unwieldy objects such as long strings, images, etc with non-uniform sizes.
2. To insert $x$ in dictionary set bit to 1 in location $h(x)$ (initially all bits are set to 0)
3. To lookup $y$ if bit in location $h(y)$ is 1 say yes, else no.
Bloom Filters

Bloom Filter:
tradeoff space for false positives

1. To insert $x$ in dictionary set bit to 1 in location $h(x)$ (initially all bits are set to 0).

2. To lookup $y$, if bit in location $h(y)$ is 1 say yes, else no.

3. No false negatives but false positives possible due to collisions.

Reducing false positives:

1. Pick $k$ hash functions $h_1, h_2, \ldots, h_k$ independently.

2. To insert $x$ for $1 \leq i \leq k$ set bit in location $h_i(x)$ in table $i$ to 1.

3. To lookup $y$, compute $h_i(y)$ for $1 \leq i \leq k$ and say yes only if each bit in the corresponding location is 1, otherwise say no. If probability of false positive for one hash function is $\alpha < 1$ then with $k$ independent hash function it is $\alpha^k$. 
Bloom Filters

**Bloom Filter:** tradeoff space for false positives

1. To insert $x$ in dictionary set *bit* to 1 in location $h(x)$ (initially all bits are set to 0)

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Take away points

1. Hashing is a powerful and important technique for dictionaries. Many practical applications.
2. Randomization fundamental to understanding hashing.
3. Good and efficient hashing possible in theory and practice with proper definitions (universal, perfect, etc).
4. Related ideas of creating a compact fingerprint/sketch for objects is very powerful in theory and practice.
Practical Issues

Hashing used typically for integers, vectors, strings etc.

- Universal hashing is defined for integers. To implement for other objects need to map objects in some fashion to integers (via representation)
- Practical methods for various important cases such as vectors, strings are studied extensively. See http://en.wikipedia.org/wiki/Universal_hashing for some pointers.
- Cryptographic hash functions have a different motivation and