CS 473: Algorithms

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Inequalities & QuickSort
w.h.p.

Lecture 8
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Outline

Slick Analysis of Randomized QuickSort

Concentration of Mass Around Mean
Markov’s Inequality
Chebyshev’s Inequality
Chernoff Bound

Randomized QuickSort: High Probability Analysis
Part I

Slick analysis of QuickSort
Recall: Randomized **QuickSort**

**Randomized QuickSort**

1. Pick a pivot element *uniformly at random* from the array.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.
Let $Q(A)$ be number of comparisons done on input array $A$:

1. For $1 \leq i < j < n$ let $R_{ij}$ be the event that rank $i$ element is compared with rank $j$ element.

2. $X_{ij}$ is the indicator random variable for $R_{ij}$. That is, $X_{ij} = 1$ if rank $i$ is compared with rank $j$ element, otherwise $0$. 

$Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}$ and hence by linearity of expectation, 

$E[Q(A)] = \sum_{1 \leq i < j \leq n} E[X_{ij}] = \sum_{1 \leq i < j \leq n} P[R_{ij}]$. 

A Slick Analysis of QuickSort

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$$Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}$$

and hence by linearity of expectation,

$$E\left[Q(A)\right] = \sum_{1 \leq i < j \leq n} E\left[X_{ij}\right] = \sum_{1 \leq i < j \leq n} \Pr\left[R_{ij}\right].$$
A Slick Analysis of QuickSort

\[ R_{ij} = \text{rank } i \text{ element is compared with rank } j \text{ element.} \]

**Question:** What is \( Pr[R_{ij}] \)?
$R_{ij} = \text{rank i element is compared with rank j element.}$

**Question:** What is $\Pr[R_{ij}]$?

With ranks: 6 4 8 1 2 3 7 5
**A Slick Analysis of QuickSort**

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**Question:** What is \( \text{Pr}[R_{ij}] \)?

<table>
<thead>
<tr>
<th>7</th>
<th>5</th>
<th>9</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>8</th>
<th>6</th>
</tr>
</thead>
</table>

With ranks: 6 4 8 1 2 3 7 5

As such, probability of comparing 5 to 8 is \( \text{Pr}[R_{4,7}] \).
A Slick Analysis of **QuickSort**

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**Question:** What is \( \Pr[R_{ij}] \)?

With ranks: 6 4 8 1 2 3 7 5

1. If pivot too small (say 3 [rank 2]). Partition and call recursively:

Decision if to compare 5 to 8 is moved to subproblem.
A Slick Analysis of QuickSort

\(R_{ij} = \text{rank } i \text{ element is compared with rank } j \text{ element.}\)

**Question:** What is \(\text{Pr}[R_{ij}]\)?

| 7 | 5 | 9 | 1 | 3 | 4 | 8 | 6 |
---|---|---|---|---|---|---|---|

With ranks: 6 4 8 1 2 3 7 5

1. If pivot too small (say 3 [rank 2]). Partition and call recursively:

\[
\begin{array}{cccccccc}
7 & 5 & 9 & 1 & 3 & 4 & 8 & 6 \\
\end{array}
\]
\[
\begin{array}{cccccccc}
1 & 3 & 7 & 5 & 9 & 4 & 8 & 6 \\
\end{array}
\]

Decision if to compare 5 to 8 is moved to subproblem.

2. If pivot too large (say 9 [rank 8]):

\[
\begin{array}{cccccccc}
7 & 5 & 9 & 1 & 3 & 4 & 8 & 6 \\
\end{array}
\]
\[
\begin{array}{cccccccc}
7 & 5 & 1 & 3 & 4 & 8 & 6 & 9 \\
\end{array}
\]

Decision if to compare 5 to 8 moved to subproblem.
A Slick Analysis of **QuickSort**

**Question:** What is $Pr[R_{i,j}]$?

![Sequence of numbers]

As such, probability of comparing 5 to 8 is $Pr[R_{4,7}]$.

1. If pivot is 5 (rank 4). Bingo!

![Before sorting]

$\Rightarrow$

![After sorting]
A Slick Analysis of **QuickSort**

**Question:** What is $Pr[R_{i,j}]$?

As such, probability of comparing 5 to 8 is $Pr[R_{4,7}]$.

1. If pivot is **5** (rank 4). Bingo!

   ![Diagram showing the effect of pivot being 5]

   $7 \ 5 \ 9 \ 1 \ 3 \ 4 \ 8 \ 6$  $\Rightarrow$  $1 \ 3 \ 4 \ 5 \ 7 \ 9 \ 8 \ 6$

2. If pivot is **8** (rank 7). Bingo!

   ![Diagram showing the effect of pivot being 8]

   $7 \ 5 \ 9 \ 1 \ 3 \ 4 \ 8 \ 6$  $\Rightarrow$  $7 \ 5 \ 1 \ 3 \ 4 \ 6 \ 8 \ 9$
A Slick Analysis of QuickSort

**Question:** What is $\Pr[R_{i,j}]$?

![Array](image)

As such, probability of comparing 5 to 8 is $\Pr[R_{4,7}]$.

1. If pivot is 5 (rank 4). Bingo!

2. If pivot is 8 (rank 7). Bingo!

3. If pivot in between the two numbers (say 6 [rank 5]):

5 and 8 will never be compared to each other.
A Slick Analysis of QuickSort

Question: What is $\Pr[R_{i,j}]$?

Conclusion: $R_{i,j}$ happens if and only if:

- $i$th or $j$th ranked element is the first pivot out of $i$th to $j$th ranked elements.
Question: What is $\Pr[R_{ij}]$?
Question: What is $\Pr[R_{ij}]$?

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$
A Slick Analysis of QuickSort

**Question:** What is $\Pr[R_{ij}]$?

**Lemma**

$$
\Pr[R_{ij}] = \frac{2}{j-i+1}.
$$

**Proof.**

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be elements of $A$ in sorted order. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

**Observation:** If pivot is chosen outside $S$ then all of $S$ either in left array or right array.

**Observation:** $a_i$ and $a_j$ separated when a pivot is chosen from $S$ for the first time. Once separated no comparison.

**Observation:** $a_i$ is compared with $a_j$ if and only if the first chosen pivot from $S$ is either $a_i$ or $a_j$. 
Lemma

\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

Proof.

Let \( a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \) be sort of \( A \). Let \( S = \{a_i, a_{i+1}, \ldots, a_j\} \)

**Observation:** \( a_i \) is compared with \( a_j \) if and only if the first chosen pivot from \( S \) is either \( a_i \) or \( a_j \).

**Observation:** Given that pivot is chosen from \( S \) the probability that it is \( a_i \) or \( a_j \) is exactly \( 2/|S| = 2/(j-i+1) \) since the pivot is chosen uniformly at random from the array.
Lemma

\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

Proof.

Let \( a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \) be sort of \( A \). Let \( S = \{a_i, a_{i+1}, \ldots, a_j\} \). Event \( E \) when first pivot from \( S \) is chosen.

Observation: Given \( E \) probability that the pivot is \( a_i \) or \( a_j \) is exactly
\[ \frac{2}{|S|} = \frac{2}{j-i+1}, \]
Lemma

\[ \Pr[R_{ij}] = \frac{2}{j - i + 1}. \]

Proof.

Let \( a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \) be sort of \( A \). Let \( S = \{a_i, a_{i+1}, \ldots, a_j\} \). Event \( E \) when first pivot from \( S \) is chosen.

Observation: Given \( E \) probability that the pivot is \( a_i \) or \( a_j \) is exactly \( 2/|S| = 2/(j - i + 1) \), i.e.

\[ \Pr[R_{ij} | E] = \frac{2}{j - i + 1}. \]
Lemma

\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

Proof.

Let \( a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n \) be sort of \( A \). Let \( S = \{a_i, a_{i+1}, \ldots, a_j\} \). Event \( E \) when first pivot from \( S \) is chosen.

**Observation:** Given \( E \) probability that the pivot is \( a_i \) or \( a_j \) is exactly

\[ 2/|S| = 2/(j-i+1), \text{ i.e.} \]

\[ \Pr[R_{ij}|E] = 2/(j-i+1). \]

Since \( \Pr[E] = 1 \), we get \( \Pr[R_{ij}] = 2/(j-i+1) \).
So far we know

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$
A Slick Analysis of QuickSort

Continued...

So far we know

\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

\[ E[Q(A)] = \sum_{1 \leq i < j \leq n} \Pr[R_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1} \]
A Slick Analysis of **QuickSort**

Continued...

So far we know

\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

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E[Q(A)] = \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1}
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A Slick Analysis of QuickSort

Continued...

So far we know

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$ 

$$E[Q(A)] = 2 \sum_{i=1}^{n-1} \sum_{i<j} \frac{1}{j-i+1}.$$
So far we know

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$  

$$E[Q(A)] = 2 \sum_{i=1}^{n-1} \sum_{i<j}^{n} \frac{1}{j-i+1} \leq 2 \sum_{i=1}^{n-1} \sum_{\Delta=2}^{n-i+1} \frac{1}{\Delta}$$
A Slick Analysis of **QuickSort**

Continued...

So far we know

\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

\[
E\left[ Q(A) \right] = 2 \sum_{i=1}^{n-1} \sum_{i<j} \frac{1}{j-i+1} \leq 2 \sum_{i=1}^{n-1} \sum_{\Delta=2}^{n-i+1} \frac{1}{\Delta} \\
\leq 2 \sum_{i=1}^{n-1} (H_{n-i+1} - 1) \leq 2 \sum_{1 \leq i < n} H_n
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\[ \Pr[R_{ij}] = \frac{2}{j-i+1}. \]

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\[
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\]

\[
\leq 2nH_n = O(n \log n)
\]
Part II

Inequalities
Massive randomness. Is not that random.

Consider flipping a fair coin \( n \) times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: \( k \) w.p. \( \binom{n}{k} \cdot \frac{1}{2^n} \).
Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

\[ \text{probability} \]
\[ n = 4 \]

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Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

![Probability distribution graph for $n=8$]
Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binomial distribution: $k$ w.p. $\binom{n}{k} \frac{1}{2^n}$. 

![Probability distribution graph for n = 16]
Consider flipping a fair coin $n$ times independently, head gives $1$, tail gives zero. How many $1$s? Binomial distribution: $k$ w.p. $\binom{n}{k}/2^n$. 

![Probability distribution graph](image-url)
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![Probability distribution graph with $n = 1024$]
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![Probability distribution graph for n = 4096]
Massive randomness.. Is not that random.

Consider flipping a fair coin $n$ times independently, head gives 1, tail gives zero. How many 1s? Binominal distribution: $k$ w.p. $\binom{n}{k}/2^n$. 

![Graph showing probability distribution for $n = 8192$.]
Massive randomness.. Is not that random.

This is known as concentration of mass. This is a very special case of the law of large numbers.
Informal statement of law of large numbers

For \( n \) large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.
Massive randomness.. Is not that random.

Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.
Massive randomness.. Is not that random.

**Intuitive conclusion**

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

Use of well known inequalities in analysis.
Randomized **QuickSort**: A possible analysis

**Analysis**

- Random variable $Q = \#\text{comparisons}$ made by randomized **QuickSort** on an array of $n$ elements.

Suppose $\Pr[Q \geq 10n \log n] \leq c$. Also we know that $Q \leq n^2$.

$$E[Q] \leq 10n \log n + (n^2 - 10n \log n)c.$$ 

**Question:** How to find $c$, or in other words bound $\Pr[Q \geq 10n \log n]$?
Randomized **QuickSort**: A possible analysis

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Randomized **QuickSort**: A possible analysis

### Analysis
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### Question:
How to find $c$, or in other words bound $\Pr[Q \geq 10n \log n]$?
Markov’s Inequality

Let $X$ be a non-negative random variable over a probability space $(\Omega, \Pr)$. For any $a > 0$,

$$\Pr[X \geq a] \leq \frac{E[X]}{a}$$
Markov’s Inequality

Let $X$ be a non-negative random variable over a probability space $(\Omega, \Pr)$. For any $a > 0$,

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Proof:

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \Pr[\omega]$$
$$\geq \sum_{\omega \in \Omega, X(\omega) \geq a} X(\omega) \Pr[\omega]$$
$$\geq a \sum_{\omega \in \Omega, X(\omega) \geq a} \Pr[\omega]$$
$$= a \Pr[X \geq a]$$
Markov’s Inequality: Proof by Picture

\[ \Pr(X \geq a) \]

\[ \text{Area} = a \cdot \Pr(X \geq a) \]

\[ \text{Area} = \sum_{x} x \cdot \Pr(X = x) = E[X] \]
Example: Balls in a bin

- \( n \) black and white balls in a bin.
- We wish to estimate the fraction of black balls. Let's say it is \( p^* \).
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- We wish to estimate the fraction of black balls. Let's say it is \( p^* \).
- An approach: Draw \( k \) balls with replacement. If \( B \) are black then output \( p = \frac{B}{k} \).
Example: Balls in a bin

- $n$ black and white balls in a bin.
- We wish to estimate the fraction of black balls. Let's say it is $p^*$.  
- An approach: Draw $k$ balls with replacement. If $B$ are black then output $p = \frac{B}{k}$.

Question

How large $k$ needs to be before our estimated value $p$ is close to $p^*$?
Example: Balls in a bin

A rough estimate through Markov’s inequality.

Lemma

For any $k \geq 1$, $\Pr[p \geq 2p^*] \leq \frac{1}{2}$
A rough estimate through Markov’s inequality.

**Lemma**

For any \( k \geq 1 \), \( \Pr[p \geq 2p^*] \leq \frac{1}{2} \)

**Proof.**

- For each \( 1 \leq i \leq k \) define random variable \( X_i \), which is 1 if \( i^{th} \) ball is black, otherwise 0.
- \( \mathbb{E}[X_i] = \Pr[X_i = 1] = p^* \).
Example: Balls in a bin

A rough estimate through Markov’s inequality.

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For any $k \geq 1$, $\Pr[p \geq 2p^*] \leq \frac{1}{2}$

Proof.

- For each $1 \leq i \leq k$ define random variable $X_i$, which is 1 if $i$\textsuperscript{th} ball is black, otherwise 0.

- $\mathbb{E}[X_i] = \Pr[X_i = 1] = p^*$.

- $B = \sum_{i=1}^{k} X_i$, then $\mathbb{E}[B] = \sum_{i=1}^{k} \mathbb{E}[X_i] = kp^*$. $p = \frac{B}{k}$. 
Example: Balls in a bin

A rough estimate through Markov’s inequality.

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For any \( k \geq 1 \), \( \Pr[p \geq 2p^*] \leq \frac{1}{2} \)

Proof.

- For each \( 1 \leq i \leq k \) define random variable \( X_i \), which is 1 if \( i^{th} \) ball is black, otherwise 0.
- \( \mathbb{E}[X_i] = \Pr[X_i = 1] = p^* \).
- \( B = \sum_{i=1}^{k} X_i \), then \( \mathbb{E}[B] = \sum_{i=1}^{k} \mathbb{E}[X_i] = kp^* \). \( p = \frac{B}{k} \).
- Markov’s inequality gives, \( \Pr[p \geq 2p^*] = \)

\[
\Pr \left[ \frac{B}{k} \geq 2p^* \right] = \Pr[B \geq 2kp^*] = \Pr[B \geq 2 \mathbb{E}[B]] \leq \frac{1}{2}
\]
Chebyshev’s Inequality: Variance

Variance

Given a random variable $X$ over probability space $(\Omega, \Pr)$, variance of $X$ is the measure of how much does it deviate from its mean value. Formally, $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
Chebyshev’s Inequality: Variance

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Given a random variable $X$ over probability space $(\Omega, \Pr)$, variance of $X$ is the measure of how much does it deviate from its mean value. Formally, $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

Intuitive Derivation

Define $Y = (X - \mathbb{E}[X])^2 = X^2 - 2X \mathbb{E}[X] + \mathbb{E}[X]^2$. 
Chebyshev’s Inequality: Variance

Variance

Given a random variable $X$ over probability space $(\Omega, \Pr)$, variance of $X$ is the measure of how much does it deviate from its mean value. Formally, $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$

Intuitive Derivation

Define $Y = (X - E[X])^2 = X^2 - 2XE[X] + E[X]^2$.

**Independence**

Random variables $X$ and $Y$ are called mutually independent if

$$
\forall x, y \in \mathbb{R}, \quad \Pr[X = x \land Y = y] = \Pr[X = x] \Pr[Y = y]
$$

**Lemma**

*If $X$ and $Y$ are independent random variables then*

$$
\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).
$$
Chebyshev’s Inequality: Variance

Independence
Random variables $X$ and $Y$ are called mutually independent if
\[ \forall x, y \in \mathbb{R}, \ Pr[X = x \land Y = y] = Pr[X = x] \cdot Pr[Y = y] \]

Lemma
If $X$ and $Y$ are independent random variables, then
\[ \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y). \]

Lemma
If $X$ and $Y$ are mutually independent, then
\[ \text{E}[XY] = \text{E}[X] \cdot \text{E}[Y]. \]
Chebyshev’s Inequality

Given $a \geq 0$, $\Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$
Chebyshev’s Inequality

Given $a \geq 0$, $\Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$

Proof.

$Y = (X - E[X])^2$ is a non-negative random variable. Apply Markov’s Inequality to $Y$ for $a^2$.

$$\Pr[Y \geq a^2] \leq \frac{\mathbb{E}[Y]}{a^2} \iff \Pr[(X - E[X])^2 \geq a^2] \leq \frac{\text{Var}(X)}{a^2} \iff \Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$$
Chebyshev’s Inequality

Given $a \geq 0$, $\Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$

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$$\Pr[Y \geq a^2] \leq \frac{\mathbb{E}[Y]}{a^2} \iff \Pr[(X - E[X])^2 \geq a^2] \leq \frac{\text{Var}(X)}{a^2} \iff \Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}(X)}{a^2}$$

$\Pr[X \leq E[X] - a] \leq \frac{\text{Var}(X)}{a^2}$ AND $\Pr[X \geq E[X] + a] \leq \frac{\text{Var}(X)}{a^2}$
Lemma

For $0 < \epsilon < 1$ and $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq \frac{1}{k\epsilon^2}$.

Proof.

Recall: $X_i$ is 1 if $i$th ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = B/k$. 
Lemma

For $0 < \epsilon < 1$ and $k \geq 1$, $Pr[|p - p^*| > \epsilon] \leq \frac{1}{k\epsilon^2}$.

Proof.

• Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = B/k$.

• $Var(X_i) = E[X_i^2] - E[X_i]^2 = E[X_i] - E[X_i]^2 = p^*(1 - p^*)$
Lemma

For $0 < \epsilon < 1$ and $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq \frac{1}{k\epsilon^2}$.

Proof.

- Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$.
  $E[X_i] = p^*$, $E[B] = kp^*$. $p = \frac{B}{k}$.
- $\text{Var}(X_i) = E[X_i^2] - E[X_i]^2 = E[X_i] - E[X_i]^2 = p^*(1 - p^*)$
- $\text{Var}(B) = \sum_i \text{Var}(X_i) = kp^*(1 - p^*)$ (Exercise)
Lemma

For $0 < \epsilon < 1$ and $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq 1/k\epsilon^2$.

Proof.

Recall: $X_i$ is $1$ if $i^{th}$ ball is black, else $0$, $B = \sum_{i=1}^{k} X_i$


$Var(X_i) = E[X_i^2] - E[X_i]^2 = E[X_i] - E[X_i]^2 = p^*(1 - p^*)$

$Var(B) = \sum_i Var(X_i) = kp^*(1 - p^*)$ (Exercise)

$$\Pr[|B/k - p^*| \geq \epsilon] = \Pr[|B - kp^*| \geq k\epsilon] \leq \frac{Var(B)}{k^2\epsilon^2} = \frac{kp^*(1-p^*)}{k^2\epsilon^2} < \frac{1}{k\epsilon^2}$$ (Chebyshev)
Lemma

Let $X_1, \ldots, X_k$ be $k$ independent random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 with probability $p_i$, and 0 with probability $(1 - p_i)$. 

Proof.

In notes!
Lemma

Let $X_1, \ldots, X_k$ be $k$ independent random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 with probability $p_i$, and 0 with probability $(1 - p_i)$. Let $X = \sum_{i=1}^{k} X_i$ and $\mu = E[X] = \sum_{i} p_i$. For any $0 < \delta < 1$, it holds that:

$$\Pr[|X - \mu| \geq \delta \mu] \leq 2e^{-\frac{\delta^2 \mu}{3}}$$
Chernoff Bound

**Lemma**

Let $X_1, \ldots, X_k$ be $k$ independent random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 with probability $p_i$, and 0 with probability $(1 - p_i)$. Let $X = \sum_{i=1}^{k} X_i$ and $\mu = E[X] = \sum_i p_i$. For any $0 < \delta < 1$, it holds that:

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$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\frac{\delta^2 \mu}{3}} \text{ and } \Pr[X \leq (1 - \delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}}$$
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Let $X_1, \ldots, X_k$ be $k$ independent random variables such that, for each $i \in [1, k]$, $X_i$ equals 1 with probability $p_i$, and 0 with probability $(1 - p_i)$. Let $X = \sum_{i=1}^{k} X_i$ and $\mu = \mathbb{E}[X] = \sum_i p_i$. For any $0 < \delta < 1$, it holds that:

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Proof.

In notes!
Lemma

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$.

Proof.

Recall: $X_i$ is 1 if $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = \frac{B}{k}$.
Lemma

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$.

Proof.

Recall: $X_i$ is 1 if the $i^{th}$ ball is black, else 0, $B = \sum_{i=1}^{k} X_i$. $E[X_i] = p^*$, $E[B] = kp^*$. $p = B/k$.

\[
\Pr[|p - p^*| \geq \epsilon] = \Pr\left[\left|\frac{B}{k} - p^*\right| \geq \epsilon\right] = \Pr[|B - kp^*| \geq k\epsilon] = \Pr[|B - kp^*| \geq \left(\frac{\epsilon}{p^*}\right)kp^*]
\]
Lemma

For any $0 < \epsilon < 1$, and $k \geq 1$, $\Pr[|p - p^*| > \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}}$.

Proof.

Recall: $X_i$ is $1$ is $i^{th}$ ball is black, else $0$, $B = \sum_{i=1}^{k} X_i$.


$$\Pr[|p - p^*| \geq \epsilon] = \Pr[|\frac{B}{k} - p^*| \geq \epsilon] = \Pr[|B - kp^*| \geq k\epsilon] = \Pr[|B - kp^*| \geq (\frac{\epsilon}{p^*})kp^*]$$

(Chebyshev) $\leq 2e^{-\frac{\epsilon^2}{3p^{2*}2}}kp^* = 2e^{-\frac{k\epsilon^2}{3p^*}}$

($p^* \leq 1$) $\leq 2e^{-\frac{k\epsilon^2}{3}}$
The problem was to estimate the fraction of black balls \( p^* \) in a bin filled with white and black balls. Our estimate was \( p = \frac{B}{k} \) instead, where out of \( k \) draws (with replacement) \( B \) balls turns out black.

**Markov’s Inequality**

For any \( k \geq 1 \), \( \Pr[p \geq 2p^*] \leq \frac{1}{2} \)
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### Markov’s Inequality

For any $k \geq 1$, \( \Pr[p \geq 2p^*] \leq \frac{1}{2} \)

### Chebyshev’s Inequality

For any $0 < \epsilon < 1$, and $k \geq 1$, \( \Pr[|p - p^*| > \epsilon] \leq \frac{1}{k\epsilon^2} \).

### Chernoff Bound

For any $0 < \epsilon < 1$, and $k \geq 1$, \( \Pr[|p - p^*| > \epsilon] \leq 2e^{-\frac{k\epsilon^2}{3}} \).
Part III

Randomized **QuickSort** (Contd.)
Randomized QuickSort: Recall

**Input:** Array $A$ of $n$ numbers. **Output:** Numbers in sorted order.

**Randomized QuickSort**

1. Pick a pivot element *uniformly at random* from $A$.
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

*Note:* On every input randomized QuickSort takes $O(n \log n)$ time in expectation. On every input it may take $\Omega(n^2)$ time with some small probability.

**Question:** With what probability it takes $O(n \log n)$ time?
Randomized **QuickSort**: Recall

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Randomized QuickSort: Recall

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**Note:** On *every* input randomized **QuickSort** takes \( O(n \log n) \) time in expectation. On *every* input it may take \( \Omega(n^2) \) time with some small probability.

**Question:** With what probability it takes \( O(n \log n) \) time?
Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.
Randomized **QuickSort**: High Probability Analysis

**Informal Statement**

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

If $n = 100$ then this gives $\Pr[Q(A) \leq 32n \ln n] \geq 0.99999$. 
Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32n \ln n] \geq 1 - \frac{1}{n^3}$.

Outline of the proof

- If depth of recursion is $k$ then $Q(A) \leq kn$.
- Prove that depth of recursion $\leq 32 \ln n$ with high probability. Which will imply the result.
Randomized **QuickSort**: High Probability Analysis

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**Outline of the proof**

- If depth of recursion is $k$ then $Q(A) \leq kn$.
- Prove that depth of recursion $\leq 32 \ln n$ with high probability. Which will imply the result.

1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability at most $1/n^4$.

2. By union bound, any of the $n$ elements participates in $> 32 \ln n$ levels with probability at most
Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

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Outline of the proof
- If depth of recursion is $k$ then $Q(A) \leq kn$.
- Prove that depth of recursion $\leq 32 \ln n$ with high probability. Which will imply the result.
  1. Focus on a single element. Prove that it “participates” in $> 32 \ln n$ levels with probability at most $1/n^4$.
  2. By union bound, any of the $n$ elements participates in $> 32 \ln n$ levels with probability at most $1/n^3$.
  3. Therefore, all elements participate in $\leq 32 \ln n$ w.p. $(1 - 1/n^3)$. 
Randomized **QuickSort**: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
Randomized **QuickSort**: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
- Fix an element $s \in A$. We will track it at each level.
- Let $S_i$ be the partition containing $s$ at $i^{th}$ level.
- $S_1 = A$ and $S_k = \{s\}$. 

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CS473  
Fall 2016  
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Randomized **QuickSort**: High Probability Analysis

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- Fix an element \( s \in A \). We will track it at each level.
- Let \( S_i \) be the partition containing \( s \) at \( i^{th} \) level.
- \( S_1 = A \) and \( S_k = \{s\} \).
- We call \( s \) lucky in \( i^{th} \) iteration, if balanced split:
  
  \[
  |S_{i+1}| \leq \frac{3}{4}|S_i| \quad \text{and} \quad |S_i \setminus S_{i+1}| \leq \frac{3}{4}|S_i|.
  \]
Randomized QuickSort: High Probability Analysis

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- If $\rho = \#\text{lucky rounds in first } k \text{ rounds}$, then $|S_k| \leq (3/4)^{\rho}n$. 
Randomized **QuickSort**: High Probability Analysis

- If $k$ levels of recursion then $kn$ comparisons.
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- If $\rho = \#\text{lucky rounds in first } k \text{ rounds}$, then $|S_k| \leq (3/4)^\rho n$.
- For $|S_k| = 1$, $\rho = 4 \ln n \geq \log_{4/3} n$ suffices.
How may rounds before $4 \ln n$ lucky rounds?

- $X_i = 1$ if $s$ is lucky in $i^{th}$ iteration.
How may rounds before $4 \ln n$ lucky rounds?

- $X_i = 1$ if $s$ is lucky in $i^{th}$ iteration.
- **Observation:** $X_1, \ldots, X_k$ are independent variables.
- $\Pr[X_i = 1] = \frac{1}{2}$ Why?

Clearly, $\rho = \sum_{i=1}^{k} X_i$. Let $\mu = E[\rho] = k/2$.

Set $k = 32 \ln n$ and $\delta = 3/4$.

$(1 - \delta) = 1/4$.

Probability of NOT getting $4 \ln n$ lucky rounds out of $32 \ln n$ rounds is,

$$\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq k/8] = \Pr[\rho \leq (1 - \delta) \mu] \quad \text{(Chernoff)}$$

$$\leq e^{-\delta^2 \mu^2} = e^{-9k/64} = e^{-\frac{9}{4} \ln n} \leq \frac{1}{n^4}.$$
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- Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$. $(1 - \delta) = \frac{1}{4}$. 

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- Set $k = 32 \ln n$ and $\delta = \frac{3}{4}$. $(1 - \delta) = \frac{1}{4}$.

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Probability of NOT getting $4 \ln n$ lucky rounds out of $32 \ln n$ rounds is,

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\Pr[\rho \leq 4 \ln n] = \Pr[\rho \leq k/8] = \Pr[\rho \leq (1 - \delta) \mu] \leq e^{-\frac{\delta^2 \mu}{2}} = e^{-\frac{9k}{64}} = e^{-4.5 \ln n} \leq \frac{1}{n^4}
$$
Randomized **QuickSort** w.h.p. Analysis

- **n** input elements. Probability that depth of recursion in **QuickSort** $> 32 \ln n$ is at most $\frac{1}{n^4} \times n = \frac{1}{n^3}$. 

Theorem: With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of **QuickSort** is $\leq 32 \ln n$. Due to $n$ comparisons in each level, with high probability, the running time of **QuickSort** is $O(n \ln n)$. 

Q: How to increase the probability?
Randomized **QuickSort** w.h.p. Analysis

- n input elements. Probability that depth of recursion in **QuickSort** > $32 \ln n$ is at most $\frac{1}{n^4} \times n = \frac{1}{n^3}$.

**Theorem**

*With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of **QuickSort** is $\leq 32 \ln n$. Due to $n$ comparisons in each level, with high probability, the running time of **QuickSort** is $O(n \ln n)$.***
Randomized QuickSort w.h.p. Analysis

- n input elements. Probability that depth of recursion in QuickSort > 32 ln n is at most \( \frac{1}{n^4} \times n = \frac{1}{n^3} \).

**Theorem**

*With high probability (i.e., \( 1 - \frac{1}{n^3} \)) the depth of the recursion of QuickSort is \( \leq 32 \ln n \). Due to n comparisons in each level, with high probability, the running time of QuickSort is \( O(n \ln n) \).*

**Q:** How to increase the probability?