Dynamic Programming on Trees

Lecture 4

September 2, 2016

What is Dynamic Programming?

Every recursion can be memoized. Automatic memoization does not help us understand whether the resulting algorithm is efficient or not.

Dynamic Programming:

A recursion that when memoized leads to an *efficient* algorithm.

Key Questions:

- Given a recursive algorithm, how do we analyze the complexity when it is memoized?
- How do we recognize whether a problem admits a dynamic programming based efficient algorithm?
- How do we further optimize time and space of a dynamic programming based algorithm?

Dynamic Programming Template

1. Come up with a recursive algorithm to solve problem
2. Understand the structure/number of the subproblems generated by recursion
3. Memoize the recursion
   - set up compact notation for subproblems
   - set up a data structure for storing subproblems
4. Iterative algorithm
   - Understand dependency graph on subproblems
   - Pick an evaluation order (any topological sort of the dependency dag)
5. Analyze time and space
6. Optimize

Dynamic Programming on Trees

Fact: Many graph optimization problems are **NP-Hard**

Fact: The same graph optimization problems are in P on trees.

Why?

Dynamic Programming on Trees

Fact: Many graph optimization problems are **NP-Hard**

Fact: The same graph optimization problems are in P on trees.

Why?

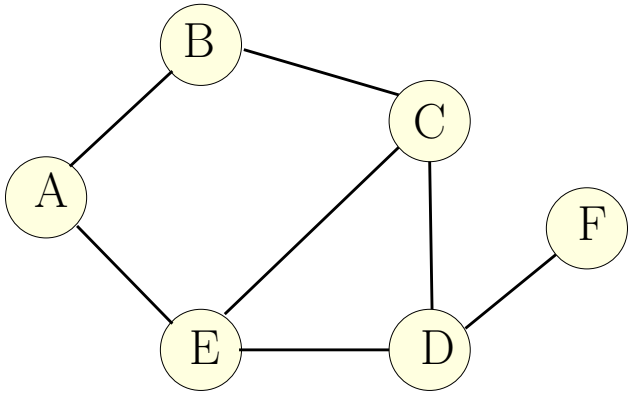
A significant reason: DP algorithm based on *decomposability*

Powerful methodology for graph algorithms via a formal notion of decomposability called **treewidth** (beyond the scope of this class)

Maximum Independent Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

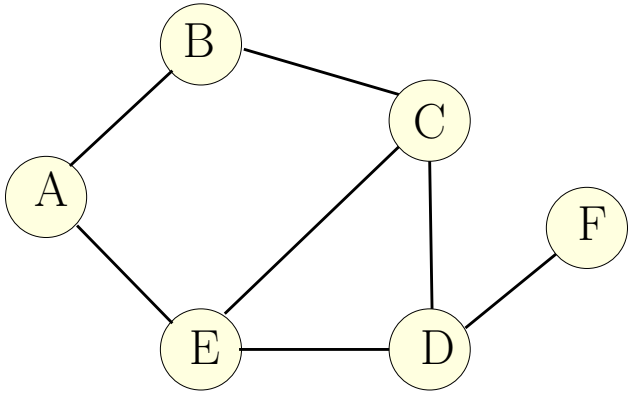


Some independent sets in graph above: $\{D\}, \{A, C\}, \{B, E, F\}$

Maximum Independent Set Problem

Input Graph $G = (V, E)$

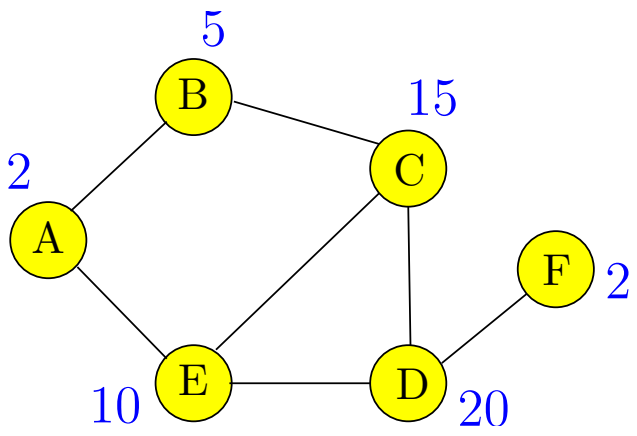
Goal Find maximum sized independent set in G



Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find maximum weight independent set in G



Maximum Weight Independent Set Problem

1. No one knows an *efficient* (polynomial time) algorithm for this problem
2. Problem is **NP-Hard** and it is *believed* that there is no polynomial time algorithm

Brute-force algorithm:

Try all subsets of vertices.

A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$.
For a vertex u let $N(u)$ be its neighbors.

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Observation

v_1: *vertex in the graph.*
One of the following two cases is true

Case 1 v_1 *is in* some *maximum independent set.*

Case 2 v_1 *is in* no *maximum independent set.*

We can try both cases *to "reduce" the size of the problem*

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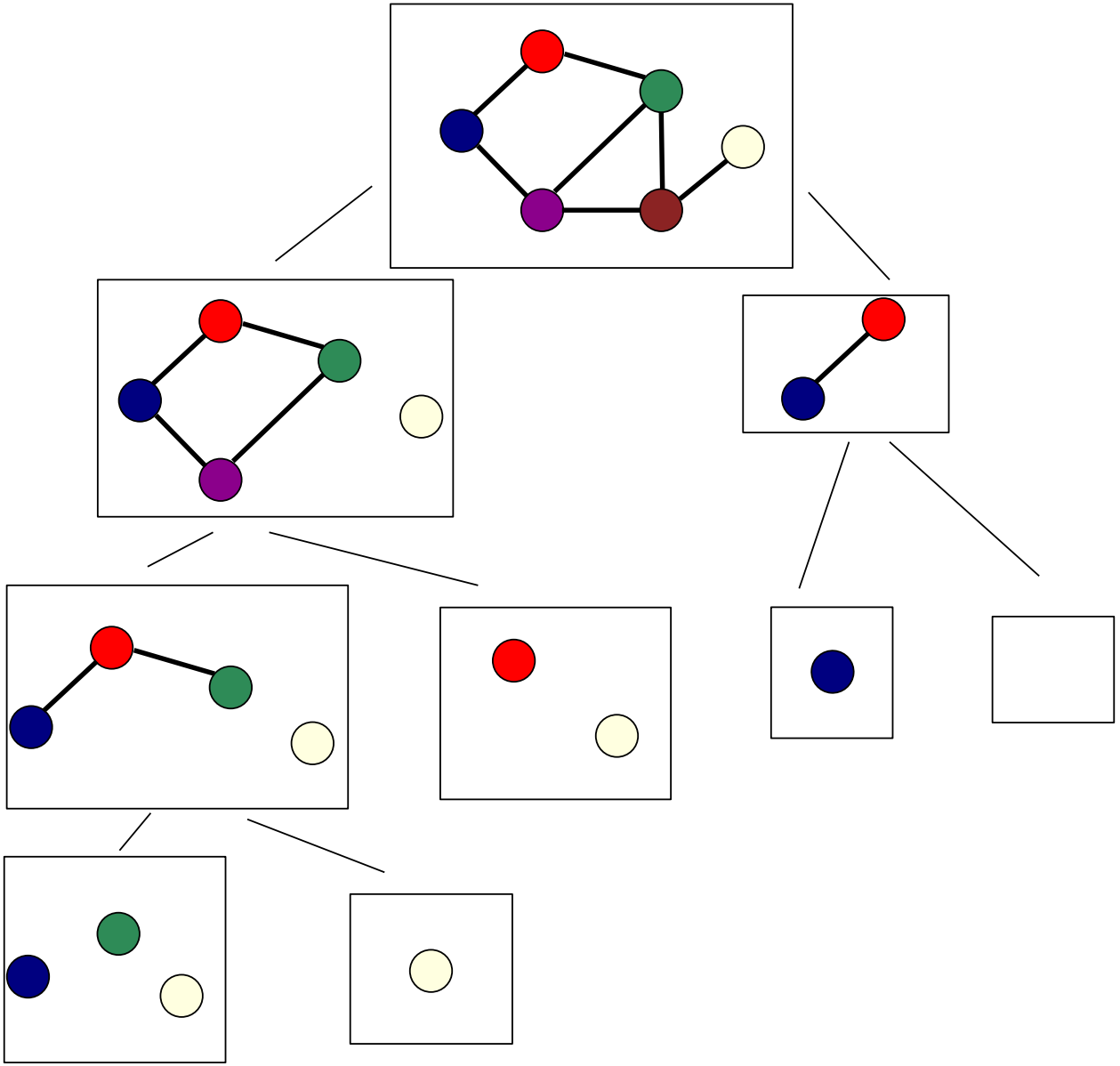
$G_1 = G - v_1$ obtained by removing v_1 and incident edges from G
$G_2 = G - v_1 - N(v_1)$ obtained by removing $N(v_1) \cup v_1$ from G

$$MIS(G) = \max\{MIS(G_1), MIS(G_2) + w(v_1)\}$$

A Recursive Algorithm

```
RecursiveMIS(G):
    if G is empty then Output 0
    a = RecursiveMIS(G − v_1)
    b = w(v_1) + RecursiveMIS(G − v_1 − N(v_1))
    Output max(a, b)
```

Example



Recursive Algorithms

..for Maximum Independent Set

Running time:

$$T(n) = T(n-1) + T\Big(n - 1 - deg(v_1)\Big) + O(1 + deg(v_1))$$

where $deg(v_1)$ is the degree of v_1. $T(0) = T(1) = 1$ is base case.

Worst case is when $deg(v_1) = 0$ when the recurrence becomes

$$T(n) = 2T(n-1) + O(1)$$

Solution to this is $T(n) = O(2^n)$.

Memoization

We can memoize the recursive algorithm.

Question: Does it lead to an efficient algorithm?

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Question: Does it lead to an efficient algorithm?

What is number of subproblems if started on graph with n nodes?

Exercise: Show that even when G is a cycle the number of subproblems is exponential in n.

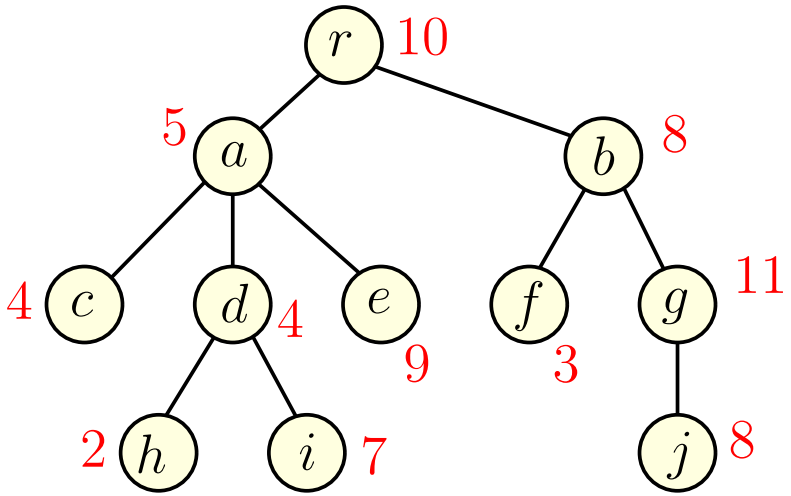
Part I

Maximum Weighted Independent Set in Trees

Maximum Weight Independent Set in a Tree

Input Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in T



Maximum weight independent set in above tree: ??

A Recursive Algorithm

For an arbitrary graph G:

1. Number vertices as $v_1, v_2, \ldots, v_n$
2. Find recursively optimum solutions without v_n (recurse on $G - v_n$) and with v_n (recurse on $G - v_n - N(v_n)$ & include v_n).
3. Saw that if graph G is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree?

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3. Saw that if graph G is arbitrary there was no good ordering that resulted in a small number of subproblems.

What about a tree? Natural candidate for v_n is root r of T?

Towards a Recursive Solution

Natural candidate for v_n is root r of T? Let $\mathcal{O}$ be an optimum solution to the whole problem.

Case $r \notin \mathcal{O}$: Then $\mathcal{O}$ contains an optimum solution for each subtree of T hanging at a child of r.

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Case $r \in \mathcal{O}$: None of the children of r can be in $\mathcal{O}$. $\mathcal{O} - \{r\}$ contains an optimum solution for each subtree of T hanging at a grandchild of r.

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Subproblems? Subtrees of T rooted at nodes in T.

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How many of them?

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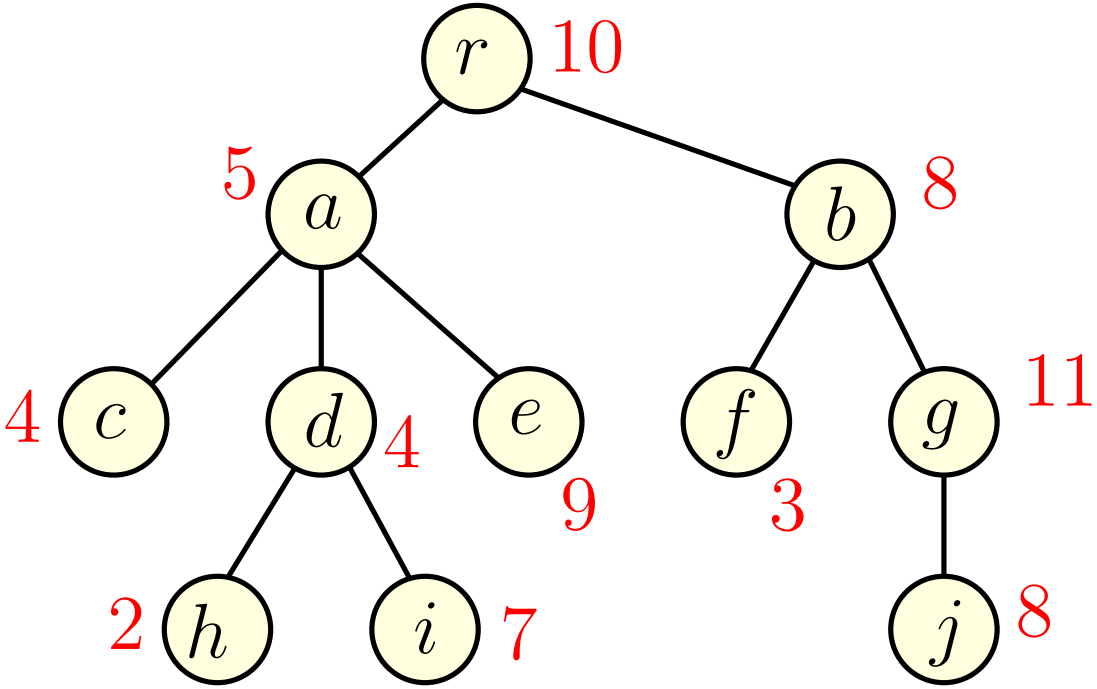
Subproblems? Subtrees of T rooted at nodes in T.

How many of them? $O(n)$

Example



A Recursive Solution

$T(u)$: subtree of T hanging at node u
$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) =$$

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$OPT(u)$: max weighted independent set value in $T(u)$

$$OPT(u) = \max \begin{cases} \sum_{v \text{ child of } u} OPT(v), \\ w(u) + \sum_{v \text{ grandchild of } u} OPT(v) \end{cases}$$

Iterative Algorithm

1. Compute $OPT(u)$ bottom up. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of u
2. What is an ordering of nodes of a tree T to achieve above?

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2. What is an ordering of nodes of a tree T to achieve above? Post-order traversal of a tree.

Iterative Algorithm

MIS-Tree(T):

Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of T

for $i = 1$ to n **do**

$$M[v_i] = \max\left(\begin{array}{c} \sum_{v_j \text{ child of } v_i} M[v_j], \\ w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \end{array}\right)$$

return $M[v_n]$ (* Note: v_n is the root of T *)

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Space:

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Space: $O(n)$ to store the value at each node of T

Running time:

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1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.

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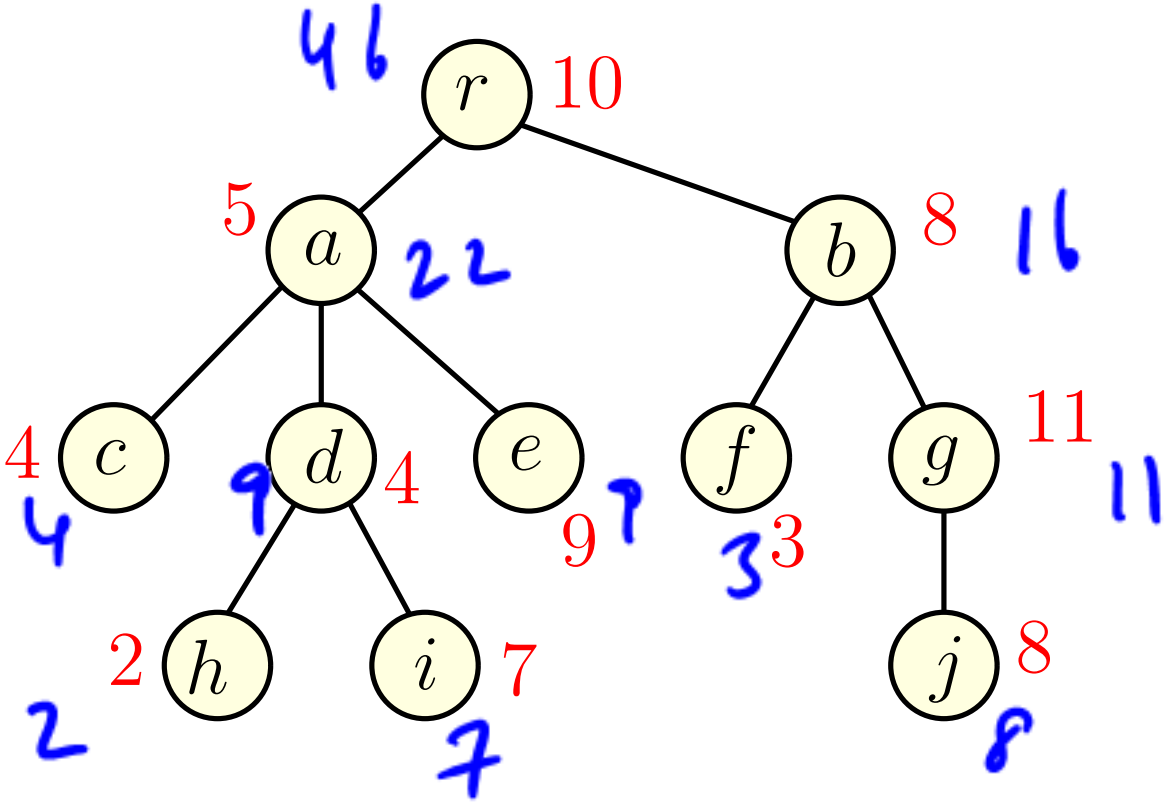
Running time:

1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.
2. Better bound: $O(n)$. A value $M[v_j]$ is accessed only by its parent and grand parent.

Example



Why did DP work on trees?

Each node (including the root) is a *separator*!

Definition

Given a graph $G = (V, E)$ a set of nodes $S \subset V$ is a *separator* for G if $G - S$ has at least two connected components.

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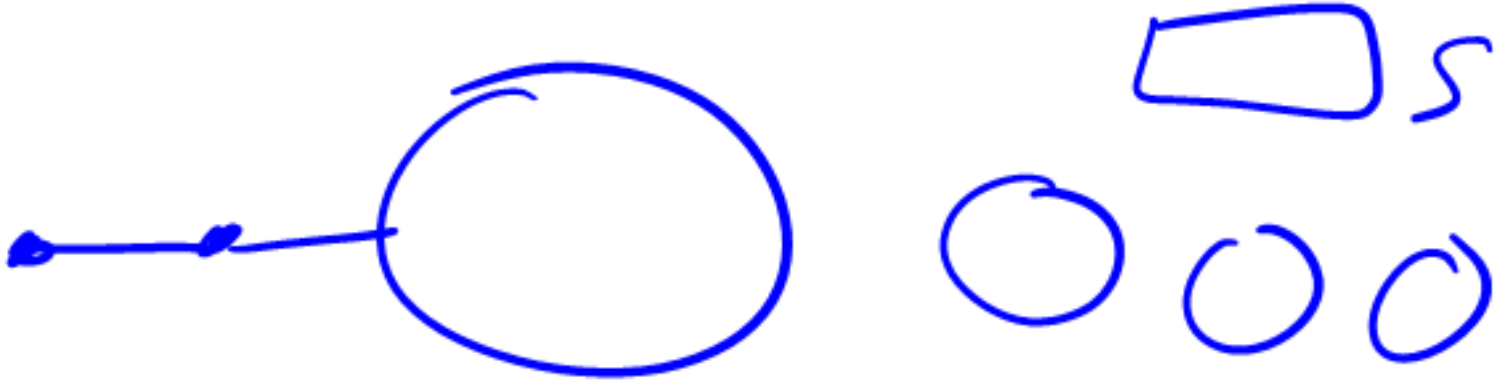
S is a *balanced* separator if each connected component of $G - S$ has at most $2|V(G)|/3$ nodes.



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Definition

S is a *balanced* separator if each connected component of $G - S$ has at most $2|V(G)|/3$ nodes.

Exercise: Prove that every tree T has a balanced separator consisting of a single node.

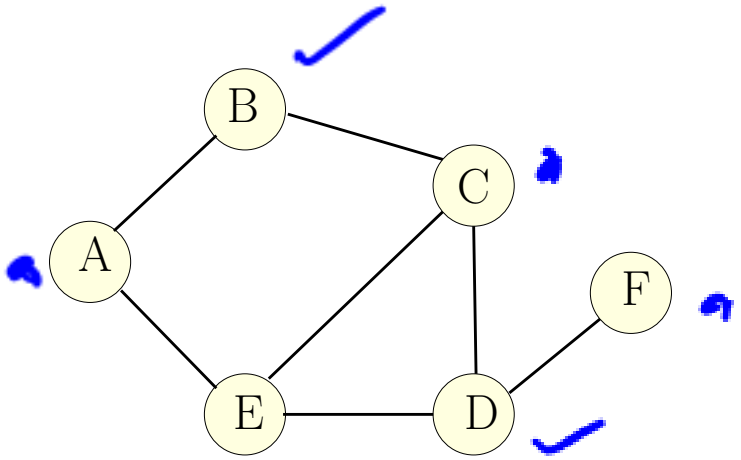
Part II

Minimum Dominating Set in Trees

Minimum Dominating Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is a dominating set if for all $v \in V$, either $v \in S$ or a neighbor of v is in S.

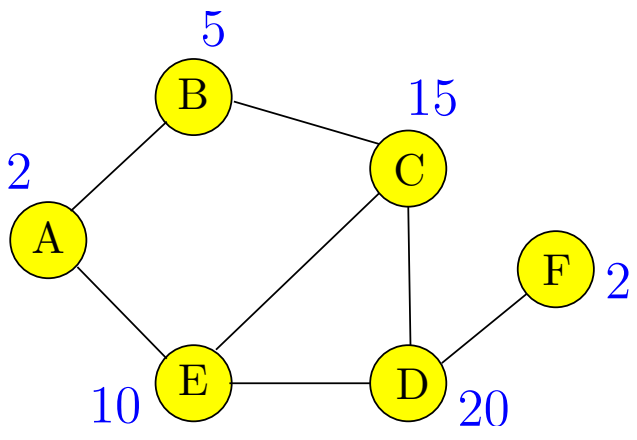


Some dominating sets in graph above: $\{A, B, C, D, E, F\}$,

Minimum Weight Dominating Set Problem

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find minimum weight dominating set in G



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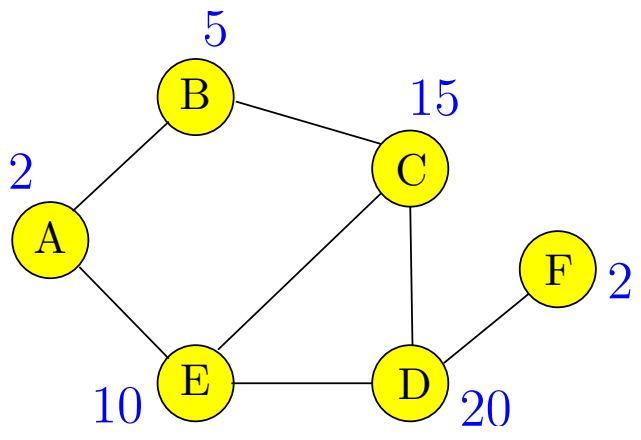
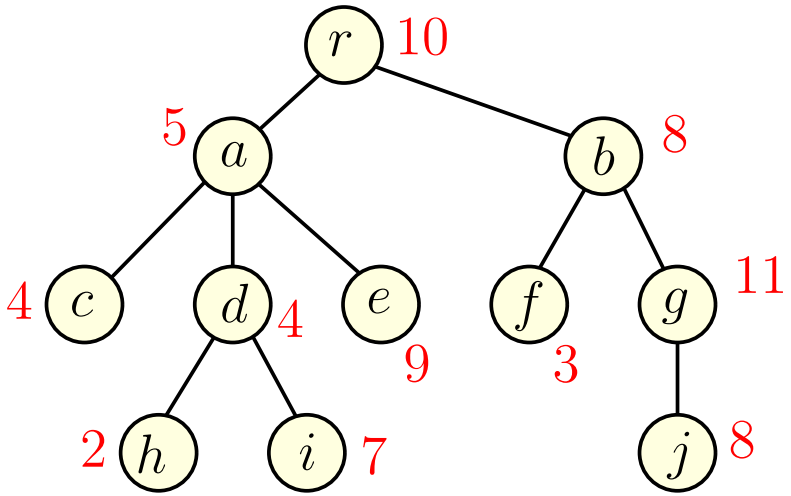


NP-Hard problem

Minimum Weight Dominating Set in a Tree

Input Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find minimum weight dominating set in T



Minimum weight dominating set in above tree: ??

Recursive Algorithm

r is root of T. Let $\mathcal{O}$ be an optimum solution for T.

Case $r \notin \mathcal{O}$: Then $\mathcal{O}$ must contain some child of r. Which one?

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Case $r \in \mathcal{O}$: None of the children of r *need* to be in $\mathcal{O}$ because r can dominate them. However, they may have to be.

Recursive Algorithm

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Case $r \in \mathcal{O}$: None of the children of r *need* to be in $\mathcal{O}$ because r can dominate them. However, they may have to be.

In both cases it is not feasible to express $|\mathcal{O}|$ easily as optimum solution values of children or descendants of r.

Removing r decomposes T into subtrees rooted at children of r.
However, not easy to decompose problem structure recursively.
Problems at children of r are *dependent*.
Need to introduce additional variable(s).

Recursive Algorithm: Understanding Dependence

Let $u_1, u_2, \ldots, u_k$ be children of root r of T

What "information" do $T_{u_1}, \ldots, T_{u_k}$ need to know about r's status in an optimum solution in order to become "independent"

Recursive Algorithm: Understanding Dependence

Let $u_1, u_2, \ldots, u_k$ be children of root r of T

What "information" do $T_{u_1}, \ldots, T_{u_k}$ need to know about r's status in an optimum solution in order to become "independent"

- Whether r is included in the solution
- If r is not included then which of the children is going to dominate it. Equivalently, T_{u_i} needs to know whether it should cover r or some other child will.

Recursive Algorithm: Introducing Variables

- u: node in tree
- pi: boolean variable to indicate whether parent is in solution. $pi = 0$ means parent is not included. $pi = 1$ means it is included.
- cp: boolean variable to indicate whether node needed to cover parent. $cp = 1$ means parent needs to be covered. $cp = 0$ means not needed.

$OPT(u, pi, cp)$: value of minimum dominating set in T_u with booleans pi and cp with meaning above.

$OPT(r, 0, 0)$: value of minimum dominating set in T
$OPT(u, 1, 0)$: value of min dominating set in T_u with parent of u included no need to cover parent.

Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of u?

Recursive Solution

Can we express $OPT(u, pi, cp)$ recursively via children of u? First consider $OPT(u, 0, 0)$ which is the value of a minimum dominating set in T_u where we assume that u's parent is not included and u does not need to cover its parent. Let C_u be children of u.

Case u is included: Then u does not need to covered by any child and we recurse on children with u being included.

$$OPT(u, 0, 0) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$$

Case u is not included: Then u needs to be covered by some child. We do a min over all children.

$$OPT(u, 0, 0) = \min_{v \in C_u}(OPT(v, 0, 1) + \sum_{v' \in C_u - v} OPT(v', 0, 0))$$

Since one of these cases has to be true, we take the min of the values in the above two cases to compute $OPT(u, 0, 0)$.

Recursive Solution

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Case u is not included: Since u's parent is included u does not need to be covered by its children. Thus we have,
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Since one of these cases has to be true, we take the min of the values in the above two cases to compute $OPT(u, 0, 0)$.

Caution: Not including u may appear to be always advantageous but it is not true.

Recursive Solution

Consider $OPT(u, 0, 1)$ which is the value of a minimum dominating set in T_u where we assume that u's parent is not included and u needs to cover its parent. Let C_u be children of u.

Case u is included: Then u does not need to covered by any child and we recurse on children with u being included.
$OPT(u, 0, 1) = w(u) + \sum_{v \in C_u} OPT(v, 1, 0)$

Case u is not included: This does not arise because u has to cover its parent.

Recursive Solution

Consider $OPT(u, 1, 1)$ which is the value of a minimum dominating set in T_u where we assume that u's parent is included and u needs to cover its parent.

This subproblem does not make sense since if u's parent is included then u does not need to cover it. In other words it suffices to only consider the subproblems $OPT(u, 0, 0), OPT(u, 1, 0), OPT(u, 0, 1)$.

Base Cases

Leaves are base cases. If u is a leaf.

- $OPT(u, 0, 0) = w(u)$
- $OPT(u, 1, 0) = 0$
- $OPT(u, 0, 1) = w(u)$

DP Algorithm

- Minimum weight dominating set value in T is $OPT(r, 0, 0)$
- To compute $OPT(r, 0, 0)$ we need to compute recursively $OPT(u, 0, 0), OPT(u, 1, 0), OPT(u, 0, 1)$ for all $u \in T$. Thus number of subproblems is $O(n)$.
- Can do this bottom up from leaves to root.

Iterative Algorithm

```
DominatingSet-Tree(T):
    Let v_1, v_2, ..., v_n be a post-order traversal of nodes of T
    Allocate array M[1..n, 0..1, 0..1] to store OPT(v_i, pi, cp) values
    for i = 1 to n do
        Compute OPT(v_i, 0, 0), OPT(v_i, 1, 0) and OPT(v_i, 0, 1) using
            values of children of v_i stored in M,
            or via base cases if v_i is leaf

            Store computed values in M for use by parent of v_i.
    return OPT(v_n, 0, 0) (* Note:  v_n is the root of T *)
```

Exercise: Work out details and prove an $O(n)$ time implementation.

Recap

- To obtain recursive solution we introduced additional variables based on "information" needed to decompose
- Decomposition depends both on structure (trees decompose via separators) and objective function
- Subproblems and recursion are almost defined hand in hand

Part III

Maximum Independent Set in Planar Graphs

Planar Graphs

Definition

(Informal): A graph $G = (V, E)$ is planar if it can be drawn in the plane without edges crossing.
(More formal): G is planar if there is a mapping of $\phi : V \to \mathbb{R}^2$ to distinct points in the 2-dimensional Euclidean plane and a mapping of each edge $uv \in E$ to a non-crossing curve $\psi(uv)$ that connects $\phi(u)$ to $\phi(v)$ such that the curves corresponding to different edges intersect only at $\phi(V)$.

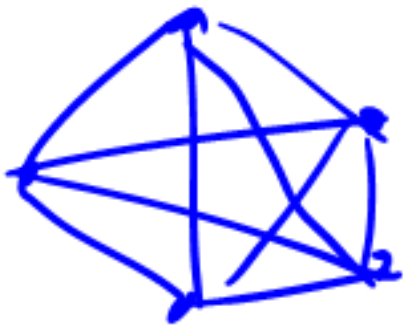
Planar graphs are very important in both theory and practice.

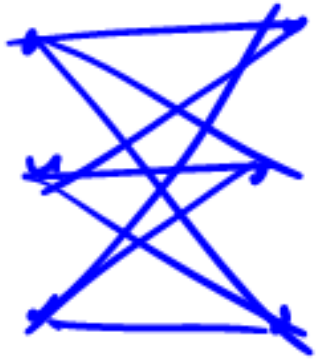
Planar Graph Theorems

Many beautiful properties and theorems.

- Euler's theorem. $|E| \leq 3|V| - 6$ for every planar graph. Hence there is always a node of degree at most 5. Thus planar graphs are 5-degenerate.
- Kuratowski's theorem: G is planar iff it "excludes" K_5 or $K_{3,3}$
- 4-Color theorem. Every planar graph is 4-colorable.
- Planar Separator Theorem: Every planar graph G on n nodes has a balance separator of size $O(\sqrt{n})$.

Properties are exploited to develop algorithms.





Maximum Independent Set in Planar Graphs

MIS in general graphs: best known algorithm runs in time $O(1.889^n)$.

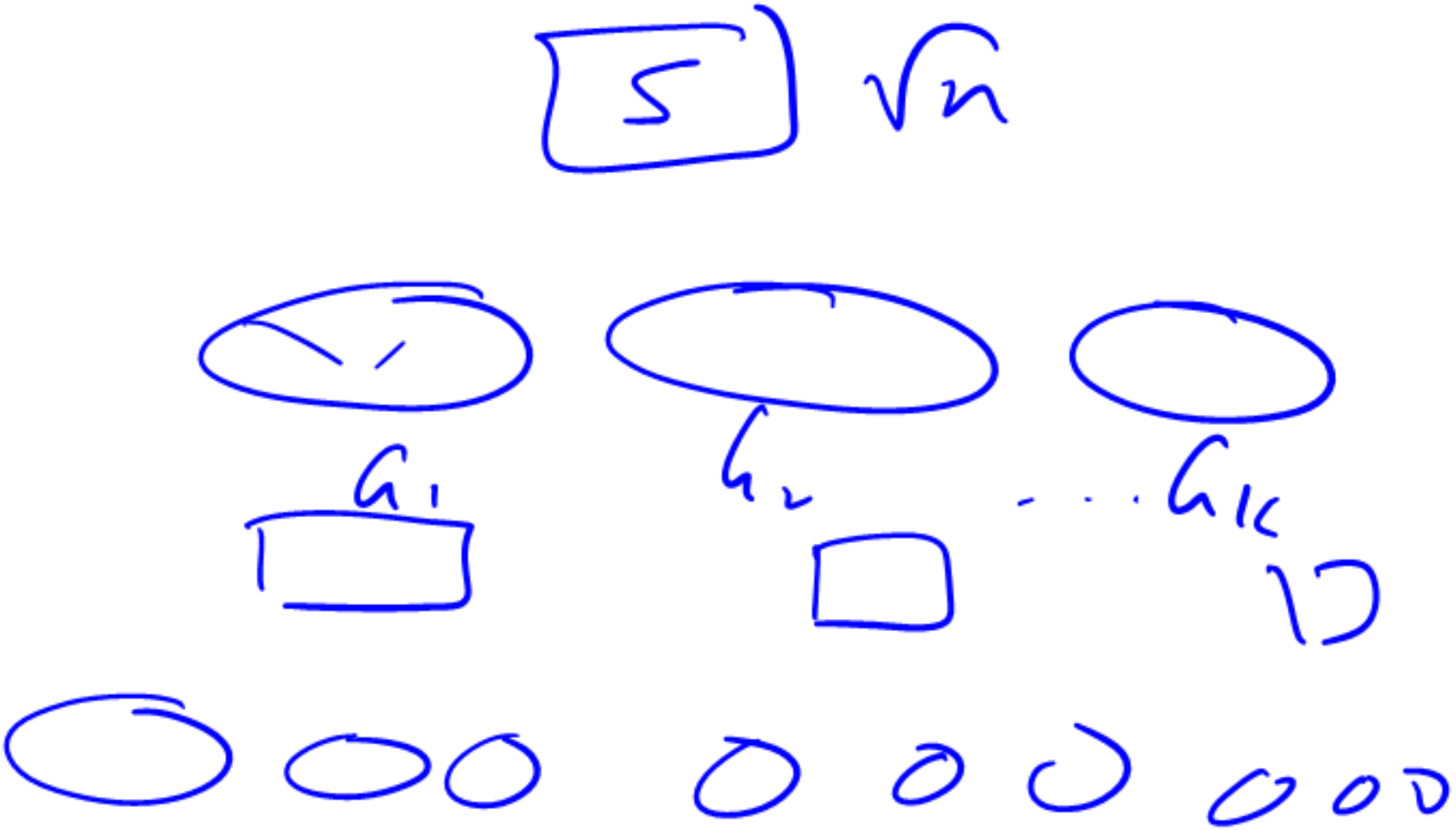
Believed that MIS in general graphs requires c^n time for some fixed $c > 1$.

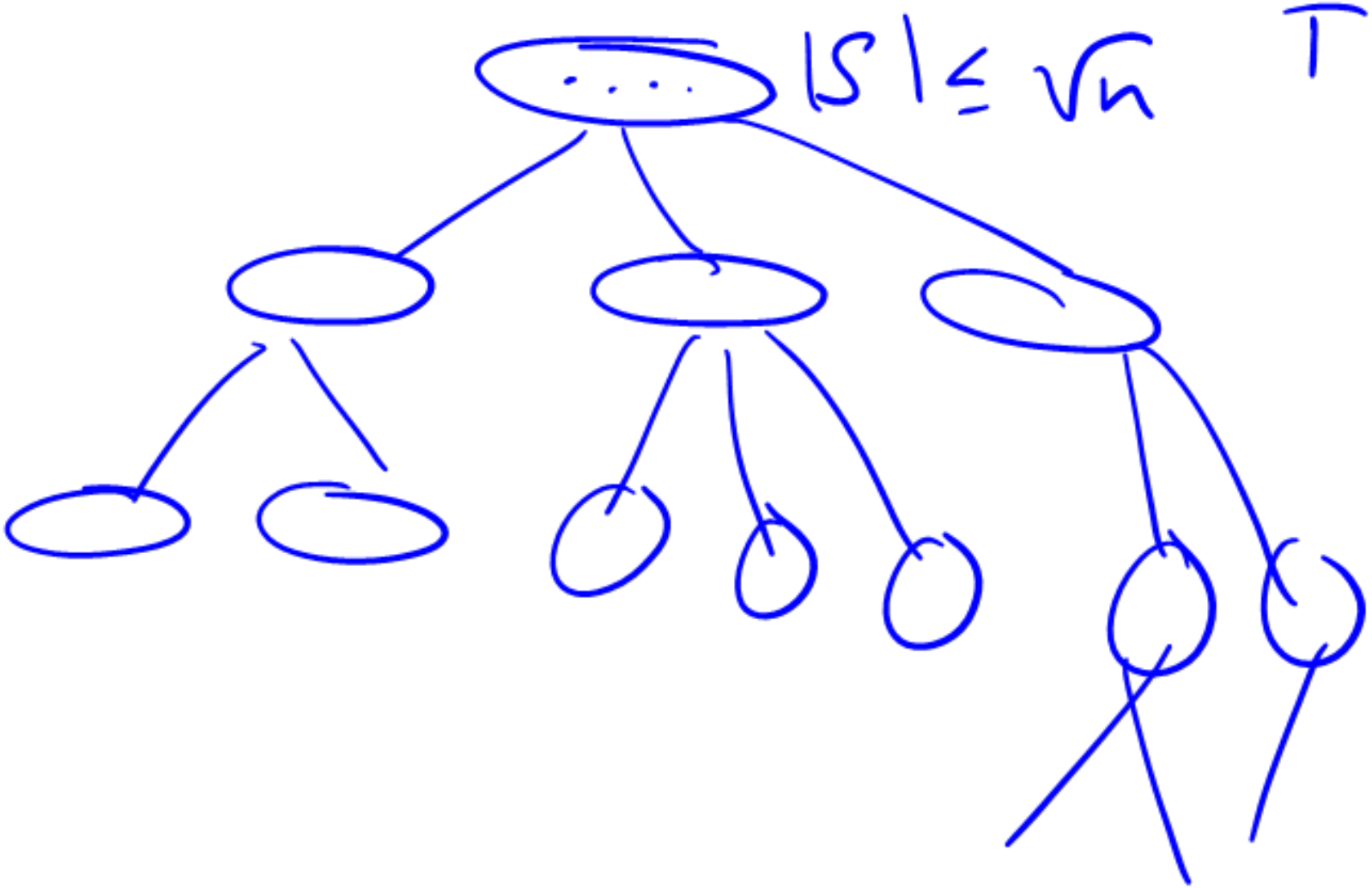
However, can solve MIS in planar graphs in $2^{O(\sqrt{n})}$ time.

Recursive Decomposition via Planar Separator Theorem

```
RecursiveDecomp(G):
    If |V(G)| ≤ n_0
        Output tree T with single root node containing V(G)
    Else
        Compute balanced separator S of G of size O(√|V(G)|)
        For each component G_i of G − S do
            T_i = RecursiveDecomp(G_i)
        Create T with root r containing S
        Make each T_i a child of r
        Output T
```

Using linear-time algorithm to compute planar separator, algorithms run time is $O(n)$.


S
√n
G1
G2
Gk


$|S| \le \sqrt{n}$
T

Properties of Decomposition Tree T

Let T be the decomposition tree for planar G with n nodes

- T is a rooted tree where each tree-node $v \in V(T)$ there is a set $S_v \subseteq V(G)$ of G's vertices associated with it. $|S_v| = O(\sqrt{n})$ for all $v \in V(T)$.
- $S_v \cap S_u = \emptyset$ for $u \neq v$ and $\cup_{v \in V(T)} S_v = V(G)$.
- For node v let $S'_v = \cup_{v \in T_v} S_v$ be the nodes in sub-tree of T rooted at v. Let $G_v = G[S'_v]$ be the sub-graph of G induced by S'_v. Then S_v is a balanced separator for G_v.
- Depth of T is $O(\log n)$. $|V(T) = O(n)$.