# Approximation Algorithms using Linear Programming

Lecture 21

November 10, 2015

# 21.1: Weighted vertex cover

# Weighted vertex cover

## Weighted Vertex Cover problem

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$.
Each vertex $\mathbf{v} \in \mathbf{V}$: cost $\mathbf{c_v}$.
Compute a vertex cover of minimum cost.

1. vertex cover: subset of vertices $\mathbf{V}$ so each edge is covered.
2. **NP-Hard**
3. ...unweighted **Vertex Cover** problem.
4. ... write as an integer program (IP):
5. $\forall \mathbf{v} \in \mathbf{V}$: $x_\mathbf{v} = 1 \iff \mathbf{v}$ in the vertex cover.
6. $\forall \mathbf{vu} \in \mathbf{E}$: covered. $\implies x_\mathbf{v} \vee x_\mathbf{u}$ **true**. $\implies x_\mathbf{v} + x_\mathbf{u} \geq 1$.
7. minimize total cost: $\min \sum_{\mathbf{v} \in \mathbf{V}} x_\mathbf{v} \mathbf{c_v}$.

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## State as IP $\implies$ Relax $\implies$ LP

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\begin{aligned}
\min \quad & \sum_{v \in V} c_v x_v, & & \\
\text{such that} \quad & x_v \in \{0,1\} & \forall v \in V & \qquad (1)\\
& x_v + x_u \geq 1 & \forall vu \in E. &
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1. ... **NP-Hard**.
2. relax the integer program.
3. allow $x_v$ get values $\in [0,1]$.
4. $x_v \in \{0,1\}$ replaced by $0 \leq x_v \leq 1$. The resulting LP is

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# Weighted vertex cover – rounding the LP

1. Optimal solution to this $\mathrm{LP}$: $\widehat{x_v}$ value of var $X_v$, $\forall v \in V$.
2. optimal value of $\mathrm{LP}$ solution is $\widehat{\alpha} = \sum_{v \in V} c_v \widehat{x_v}$.
3. optimal integer solution: $x_v^I$, $\forall v \in V$ and $\alpha^I$.
4. **Any valid solution to IP is valid solution for LP!**
5. $\widehat{\alpha} \leq \alpha^I$.
   Integral solution not better than $\mathrm{LP}$.
6. Got fractional solution (i.e., values of $\widehat{x_v}$).
7. Fractional solution is better than the optimal cost.
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# How to round?

1. consider vertex **v** and fractional value $\widehat{x_{\mathsf{v}}}$.
2. If $\widehat{x_{\mathsf{v}}} = 1$ then include in solution!
3. If $\widehat{x_{\mathsf{v}}} = 0$ then do include in solution.
4. if $\widehat{x_{\mathsf{v}}} = 0.9 \implies \mathrm{LP}$ considers **v** as being $0.9$ useful.
5. The $\mathrm{LP}$ puts its money where its belief is...
6. ...$\widehat{\alpha}$ value is a function of this "belief" generated by the $\mathrm{LP}$.
7. Big idea: Trust $\mathrm{LP}$ values as guidance to usefulness of vertices.
8. Pick all vertices $\geq$ threshold of usefulness according to $\mathrm{LP}$.
9. $S = \left\{ \mathsf{v} \;\middle|\; \widehat{x_{\mathsf{v}}} \geq 1/2 \right\}$.
10. **Claim**: $S$ a valid vertex cover, and cost is low.
11. Indeed, edge cover as: $\forall \mathbf{vu} \in \mathbf{E}$ have $\widehat{x_{\mathsf{v}}} + \widehat{x_{\mathsf{u}}} \geq 1$.
12. $\widehat{x_{\mathsf{v}}}, \widehat{x_{\mathsf{u}}} \in (0,1) \implies \widehat{x_{\mathsf{v}}} \geq 1/2$ or $\widehat{x_{\mathsf{u}}} \geq 1/2$.
    $\implies \mathbf{v} \in S$ or $\mathbf{u} \in S$ (or both).
    $\implies S$ covers all the edges of $\mathbf{G}$.

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9. $S = \left\{ \mathsf{v} \;\middle|\; \widehat{x_{\mathsf{v}}} \geq 1/2 \right\}$.
10. **Claim**: $S$ a valid vertex cover, and cost is low.
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since $\widehat{x_\mathsf{v}} \geq 1/2$ as $\mathsf{v} \in S$.

$\alpha^I$ is cost of the optimal solution $\implies$

**Theorem**

*The* **Weighted Vertex Cover** *problem can be* $2$*-approximated by solving a single* LP*. Assuming computing the* LP *takes polynomial time, the resulting approximation algorithm takes polynomial time.*

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# The lessons we can take away

## Or not - boring, boring, boring.

1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
2. Not aware of any other $2$-approximation algorithm does not use $\mathrm{LP}$. (For the weighted case!)
3. Solving a **relaxation** of an optimization problem into a $\mathrm{LP}$ provides us with insight.
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# 21.2: Revisiting Set Cover

# Revisiting **Set Cover**

1. Purpose: See new technique for an approximation algorithm.
2. Not better than greedy algorithm already seen $O(\log n)$ approximation.

### Set Cover

**Instance**: $(S, \mathcal{F})$

$S$: set of $n$ elements

$\mathcal{F}$: family of subsets of $S$, s.t. $\bigcup_{X \in \mathcal{F}} X = S$.

**Question:** The set $\mathcal{X} \subseteq \mathcal{F}$ such that $\mathcal{X}$ contains as few sets as possible, and $\mathcal{X}$ covers $S$.

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# Set Cover – IP & LP

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\begin{aligned}
\min \quad & \alpha = \sum_{U \in \mathfrak{F}} x_U, & \\
\text{s.t.} \quad & x_U \in \{0, 1\} & \forall U \in \mathfrak{F}, \\
& \sum_{U \in \mathfrak{F}, s \in U} x_U \geq 1 & \forall s \in S.
\end{aligned}
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Next, we relax this IP into the following LP.

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2. Opt IP solution: $\forall U \in \mathcal{F}$, $x_U^I$, and $\alpha^I$.
3. Use LP solution to guide in rounding process.
4. If $\widehat{x_U}$ is close to $1$ then pick $U$ to cover.
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6. Idea: Pick $U \in \mathcal{F}$: randomly choose $U$ with **probability** $\widehat{x_U}$.
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# Set Cover – Rounding continued

1. **Sol**: Repeat rounding $m = 10 \lceil \lg n \rceil = O(\log n)$ times.
2. $n = |S|$.
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1. For an element $s \in S$, we have that
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# Cost of solution

1. Have: $\mathbf{E}\left[\text{cost of } \mathcal{G}_i\right] \leq \alpha^I$.
2. $\implies$ Each iteration expected cost of cover $\leq$ cost of optimal solution (i.e., $\alpha^I$).
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# The result

## Theorem

*By solving an* $\mathrm{LP}$ *one can get an* $O(\log n)$*-approximation to set cover by a randomized algorithm. The algorithm succeeds with high probability.*

# Same algorithms works for...

## Corollary

*By solving an* LP *one can get an* $O(\log n)$*-approximation to set cover by a randomized algorithm. The algorithm also works for the weighted case.*

$$
\begin{aligned}
\min \quad & \alpha = \sum_{U \in \mathcal{F}} w_U x_U \\
& 0 \leq x_U \leq 1 && \forall U \in \mathcal{F}, \\
& \sum_{U \in \mathcal{F}, s \in U} x_U \geq 1 && \forall s \in S.
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Rounding algorithm as before...

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# Cost of solution (weighted case)...

Same same, not the same.

1. Fractional LP solution. Target: $\widehat{\alpha}$ $\forall U \in \mathfrak{F}$: $\widehat{x_U} \in [0,1]$.
2. Integral opt solution. Target: $\alpha^I$. $\forall U \in \mathfrak{F}$: $x_U^I \in \{0,1\}$.
3. $\alpha^I = \sum_{U \in \mathfrak{F}} w_U x_U^I$.
4. Rounding. $\forall U \in \mathfrak{F}$: $\Pr[X_U = 1] = \widehat{x_U}$.
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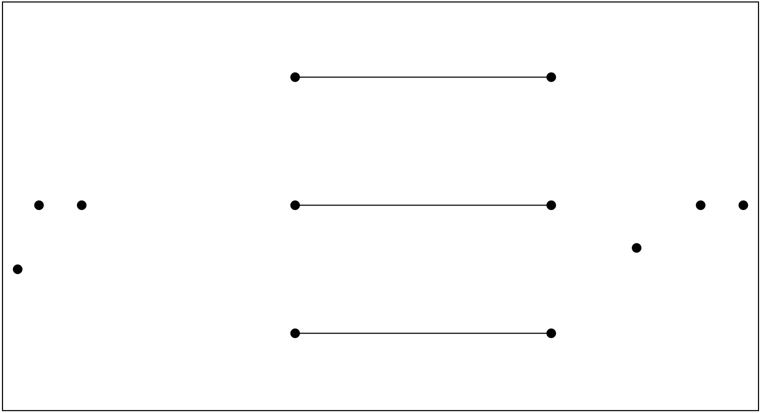
# 21.3: Minimizing congestion

# Minimizing congestion by example



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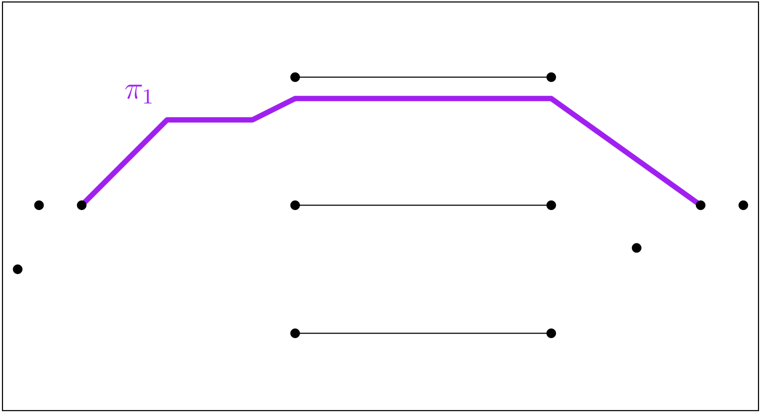

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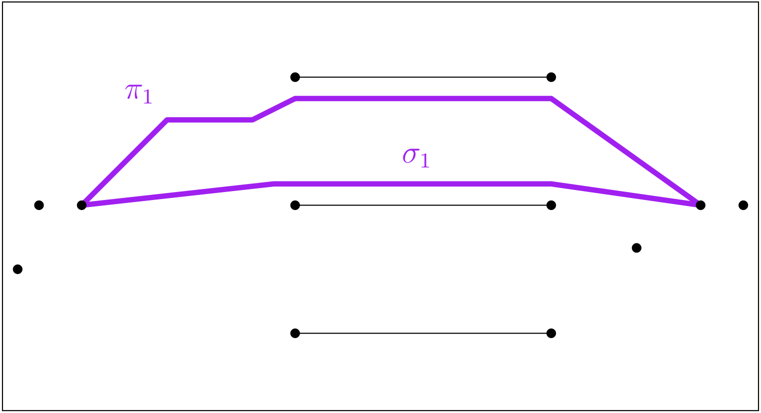

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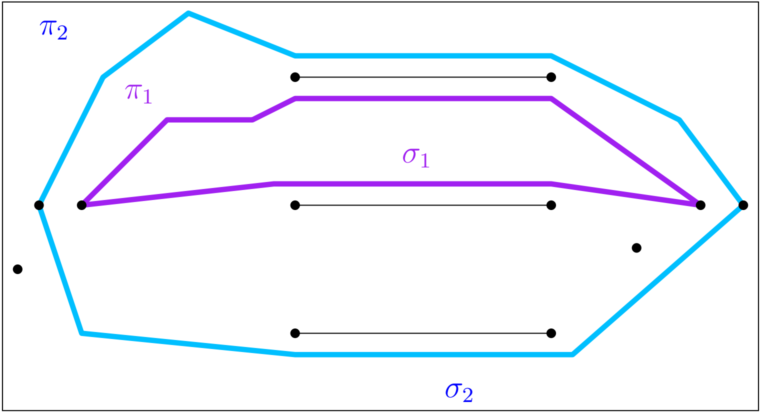

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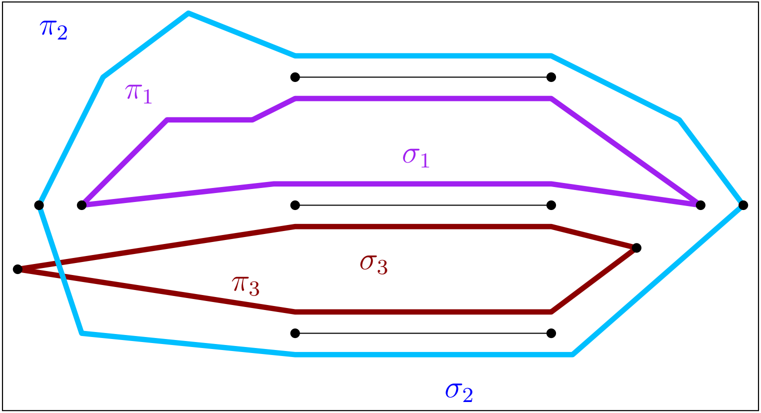

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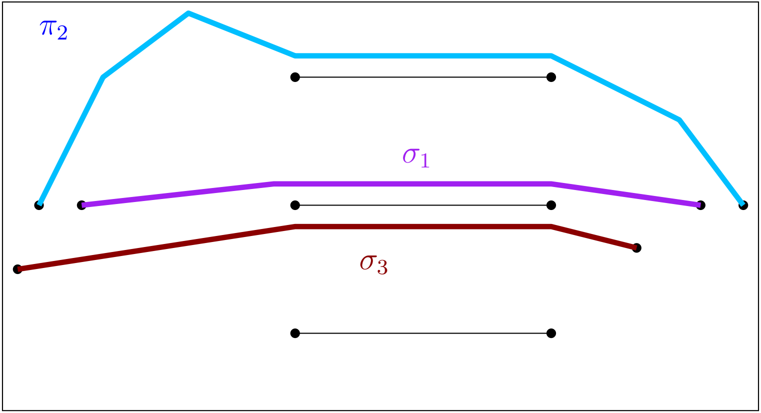

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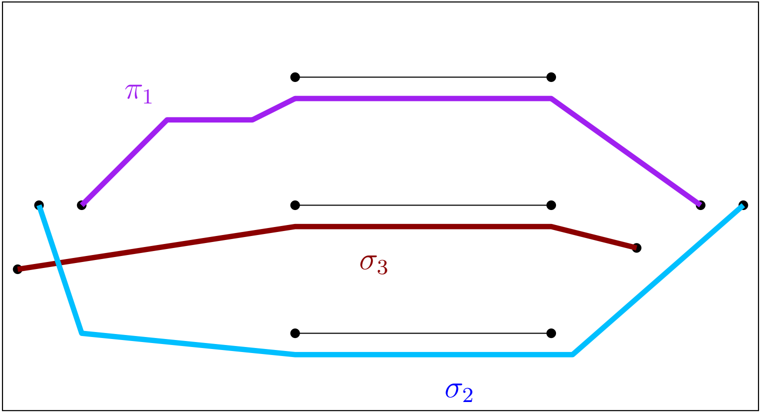

# Minimizing congestion

1. $\mathbf{G}$: graph. $n$ vertices.
2. $\pi_i$, $\sigma_i$ paths with the same endpoints $\mathbf{v}_i, \mathbf{u}_i \in \mathbf{V}(\mathbf{G})$, for $i = 1, \ldots, t$.
3. Rule I: Send one unit of flow from $\mathbf{v}_i$ to $\mathbf{u}_i$.
4. Rule II: Choose whether to use $\pi_i$ or $\sigma_i$.
5. Target: No edge in $\mathbf{G}$ is being used too much.

## Definition

Given a set $X$ of paths in a graph $\mathbf{G}$, the **congestion** of $X$ is the maximum number of paths in $X$ that use the same edge.

# Minimizing congestion

1. IP $\implies$ LP:

$$\begin{aligned} \min \quad & w \\ \text{s.t.} \quad & x_i \geq 0 && i = 1, \ldots, t, \\ & x_i \leq 1 && i = 1, \ldots, t, \\ & \sum_{\mathbf{e} \in \pi_i} x_i + \sum_{\mathbf{e} \in \sigma_i} (1 - x_i) \leq w && \forall \mathbf{e} \in E. \end{aligned}$$

2. $\widehat{x}_i$: value of $x_i$ in the optimal LP solution.
3. $\widehat{w}$: value of $w$ in LP solution.
4. Optimal congestion must be bigger than $\widehat{w}$.
5. $X_i$: random variable one with probability $\widehat{x}_i$, and zero otherwise.
6. If $X_i = 1$ then use $\pi$ to route from $\mathbf{v}_i$ to $\mathbf{u}_i$.
7. Otherwise use $\sigma_i$.

# Minimizing congestion

1. Congestion of **e** is $Y_e = \sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)$.
2. And in expectation

$$\alpha_e = \mathbb{E}\left[Y_e\right] = \mathbb{E}\left[\sum_{e \in \pi_i} X_i + \sum_{e \in \sigma_i} (1 - X_i)\right]$$
$$= \sum_{e \in \pi_i} \mathbb{E}\left[X_i\right] + \sum_{e \in \sigma_i} \mathbb{E}\left[(1 - X_i)\right]$$
$$= \sum_{e \in \pi_i} \widehat{x}_i + \sum_{e \in \sigma_i} (1 - \widehat{x}_i) \leq \widehat{w}.$$

3. $\widehat{w}$: Fractional congestion (from LP solution).

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2. $Y_{\mathrm{e}}$ is just a sum of independent $0/1$ random variables!
3. Chernoff inequality tells us sum can not be too far from expectation!

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# Minimizing congestion - continued

1. By Chernoff inequality:

$$\mathbf{Pr}\Big[Y_e \geq (1+\delta)\alpha_e\Big] \leq \exp\left(-\frac{\alpha_e \delta^2}{4}\right) \leq \exp\left(-\frac{\widehat{w}\delta^2}{4}\right).$$

2. Let $\delta = \sqrt{\frac{400}{\widehat{w}} \ln t}$. We have that

$$\Pr\Big[Y_e \geq (1+\delta)\alpha_e\Big] \leq \exp\left(-\frac{\delta^2 \widehat{w}}{4}\right) \leq \frac{1}{t^{100}},$$

3. If $t \geq n^{1/50} \implies \forall$ edges in graph congestion $\leq (1+\delta)\widehat{w}$.
4. $t$: Number of pairs, $n$: Number of vertices in **G**.

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2. Play with the numbers. If $t = n$, and $\widehat{w} \geq \sqrt{n}$. Then, the solution has congestion larger than the optimal solution by a factor of

$$1 + \delta = 1 + \sqrt{\frac{20}{\widehat{w}} \ln t} \leq 1 + \frac{\sqrt{20 \ln n}}{n^{1/4}},$$

which is of course extremely close to $1$, if $n$ is sufficiently large.

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# Minimizing congestion: result

## Theorem

1. **G**: *Graph* $n$ *vertices.*
2. $(s_1, t_1), \ldots, (s_t, t_t)$: *pairs o vertices*
3. $\pi_i, \sigma_i$: *two different paths connecting* $s_i$ *to* $t_i$
4. $\widehat{w}$: *Fractional congestion at least* $n^{1/2}$.
5. $\mathrm{opt}$: *Congestion of optimal solution.*
6. $\implies$ *In polynomial time (*$\mathrm{LP}$ *solving time) choose paths*
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# When the congestion is low

1. Assume $\widehat{w}$ is a constant.
2. Can get a better bound by using the Chernoff inequality in its more general form.
3. set $\delta = c \ln t / \ln \ln t$, where $c$ is a constant. For $\mu = \alpha_e$, we have that

$$\Pr\left[Y_e \geq (1+\delta)\mu\right] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu = \exp\Big(\mu\big(\delta - (1+\delta)\ln(1+\delta)\big)\Big) = \exp\Big(-\mu c' \ln t\Big) \leq \frac{1}{t^{O(1)}},$$

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1. Just proved that...
2. if the optimal congestion is $O(1)$, then...
3. algorithm outputs a solution with congestion $O(\log t / \log \log t)$, and this holds with high probability.

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# 21.4: Reminder about Chernoff inequality

# 21.4.1: The Chernoff Bound — General Case

# Chernoff inequality

## Problem

*Let $X_1, \ldots X_n$ be $n$ independent Bernoulli trials, where*

$$\Pr\left[X_i = 1\right] = p_i, \qquad \Pr\left[X_i = 0\right] = 1 - p_i,$$

$$Y = \sum_i X_i, \qquad \text{and} \qquad \mu = \mathbf{E}\left[Y\right].$$

*We are interested in bounding the probability that $Y \geq (1 + \delta)\mu$.*

# Chernoff inequality

## Theorem (Chernoff inequality)

*For any* $\delta > 0$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

*Or in a more simplified form, for any* $\delta \leq 2e - 1$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \exp\left(-\mu\delta^2/4\right),$$

*and*

$$\Pr\left[Y > (1+\delta)\mu\right] < 2^{-\mu(1+\delta)},$$

*for* $\delta \geq 2e - 1$.

# More Chernoff...

## Theorem

*Under the same assumptions as the theorem above, we have*

$$\mathbf{Pr}\Big[Y < (1-\delta)\mu\Big] \leq \exp\left(-\mu\frac{\delta^2}{2}\right).$$

# Notes

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