Network flow, duality and Linear Programming

Lecture 20

November 5, 2015

20.1: Network flow via linear programming

20.1.1: Network flow: Problem definition

Network flow

1. Transfer as much "merchandise" as possible from one point to another.
2. Wireless network, transfer a large file from s to t.
3. Limited capacities.

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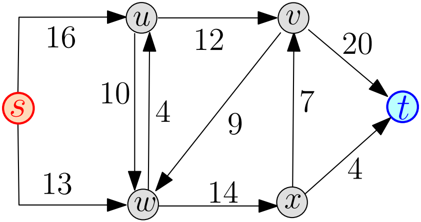

Network: Definition

1. Given a network with capacities on each connection.
2. Q: How much "flow" can transfer from source s to a sink t?
3. The flow is **splitable**.
4. Network examples: water pipes moving water. Electricity network.
5. Internet is packet base, so not quite splitable.

Definition

- ⋆ $\mathsf{G} = (\mathsf{V}, \mathsf{E})$: a **directed** graph.
- ⋆ $\forall (u, v) \in \mathsf{E}(\mathsf{G})$: **capacity** $c(u, v) \geq 0$,
- ⋆ $(u, v) \notin G \implies c(u, v) = 0$.
- ⋆ s: **source** vertex, t: target **sink** vertex.
- ⋆ G, s, t and $c(\cdot)$: form **flow network** or **network**.

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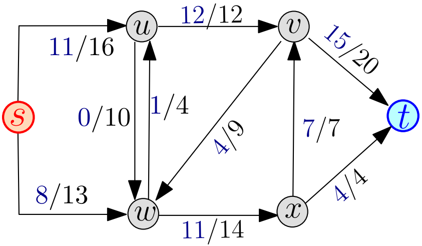

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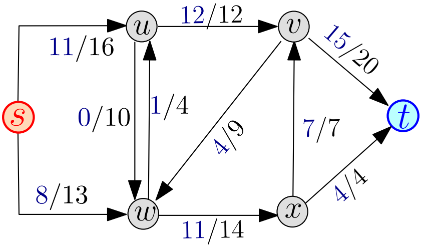

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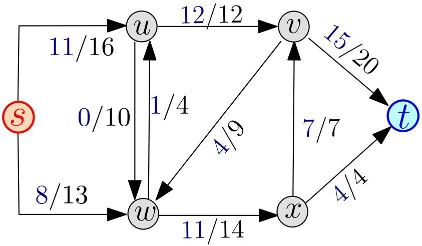

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Flow definition

Definition (flow)

flow in network is a function $f(\cdot,\cdot) : \mathbf{E}(\mathbf{G}) \rightarrow \mathbb{R}$:

(A) **Bounded by capacity**:
$\forall (u,v) \in \mathbf{E} \qquad f(u,v) \leq c(u,v).$

(B) **Anti symmetry**:
$\forall u, v \qquad f(u,v) = -f(v,u).$

(C) Two special vertices: (i) the **source** s and the **sink** t.

(D) **Conservation of flow** (Kirchhoff's Current Law):
$$\forall u \in \mathbf{V} \setminus \{s,t\} \qquad \sum_{v} f(u,v) = 0.$$

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Problem: Max Flow

1. Flow on edge can be negative (i.e., positive flow on edge in other direction).

Problem (Maximum flow)

Given a network $\mathbf{G}$ *find the* **maximum flow** *in* $\mathbf{G}$. *Namely, compute a legal flow* f *such that* $|f|$ *is maximized.*

20.1.2: Network flow via linear programming

Network flow via linear programming

Input: $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with source $\mathbf{s}$ and sink $\mathbf{t}$, and capacities $\mathbf{c}(\cdot)$ on the edges. Compute max flow in $\mathbf{G}$.

$$\forall (u,v) \in E \qquad 0 \leq x_{u \to v}$$

$$x_{u \to v} \leq \mathbf{c}(u \to v)$$

$$\forall v \in V \setminus \{\mathbf{s}, \mathbf{t}\} \qquad \sum_{(u,v) \in E} x_{u \to v} - \sum_{(v,w) \in E} x_{v \to w} \leq 0$$

$$\sum_{(u,v) \in E} x_{u \to v} - \sum_{(v,w) \in E} x_{v \to w} \geq 0$$

maximizing $\sum_{(\mathbf{s},u) \in E} x_{\mathbf{s} \to u}$

20.1.3: Min-Cost Network flow via linear programming

Min cost flow

Input:

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$: directed graph.

$\mathbf{s}$: source.

$\mathbf{t}$: sink

$\mathbf{c}(\cdot)$: capacities on edges,

ϕ: Desired amount (**value**) of flow.

$\kappa(\cdot)$: Cost on the edges.

Definition - cost of flow

cost of flow $\mathbf{f}$: $\mathrm{cost}(\mathbf{f}) = \sum_{e \in E} \kappa(e) * \mathbf{f}(e)$.

Min cost flow problem

Min-cost flow

minimum-cost s-t flow problem: compute the flow $\mathbf{f}$ of min cost that has value ϕ.

min-cost circulation problem

Instead of ϕ we have lower-bound $\ell(\cdot)$ on edges.
(All flow that enters must leave.)

Claim

If we can solve min-cost circulation $\implies$ can solve min-cost flow.

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20.2: Duality and Linear Programming

Duality...

1. Every linear program L has a **dual linear program** L'.
2. Solving the dual problem is essentially equivalent to solving the **primal linear program** original LP.
3. Lets look an example..

20.2.1: Duality by Example

Duality by Example

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

1. η: maximal possible value of target function.
2. Any feasible solution $\Rightarrow$ a lower bound on η.
3. In above: $x_1 = 1, x_2 = x_3 = 0$ is feasible, and implies $z = 4$ and thus $\eta \geq 4$.
4. $x_1 = x_2 = 0, x_3 = 3$ is feasible $\implies \eta \geq z = 9$.
5. How close this solution is to opt? (i.e., η)
6. If very close to optimal – might be good enough. Maybe stop?

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1. Add the first inequality (multiplied by 2) to the second inequality (multiplied by 3):

$$\begin{aligned} 2(\ x_1 + 4x_2 \qquad) &\le 2(1) \\ +3(3x_1 - x_2 + x_3) &\le 3(3). \end{aligned}$$

2. The resulting inequality is

$$11x_1 + 5x_2 + 3x_3 \le 11. \quad (1)$$

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1. got $11x_1 + 5x_2 + 3x_3 \le 11$.
2. inequality must hold for any feasible solution of L.
3. Objective: $z = 4x_1 + x_2 + 3x_3$ and $x_{1,}x_2$ and x_3 are all non-negative.
4. Inequality above has larger coefficients than objective (for corresponding variables)
5. For any feasible solution: $z = 4x_1 + x_2 + 3x_3 \le 11x_1 + 5x_2 + 3x_3 \le 11$,

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1. For any feasible solution:
   $z = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11,$
2. Opt solution is LP L is somewhere between 9 and 11.
3. Multiply first inequality by y_1, second inequality by y_2 and add them up:

$$
\begin{array}{cccccc}
y_1(x_1 & + & 4x_2 & &) \leq & y_1(1) \\
+\, y_2(3x_1 & - & x_2 & + \quad x_3 &) \leq & y_2(3) \\
\hline
(y_1 + 3y_2)x_1 & + & (4y_1 - y_2)x_2 & + \quad y_2 x_3 & \leq & y_1 + 3y_2.
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$$\begin{array}{ccccccc} y_1(x_1 & + & 4x_2 & & &) \le & y_1(1) \\ + \; y_2(3x_1 & - & x_2 & + & x_3 &) \le & y_2(3) \\ \hline (y_1 + 3y_2)x_1 & + & (4y_1 - y_2)x_2 & + & y_2 x_3 & \le & y_1 + 3y_2. \end{array}$$

Duality by Example: IV

$$
\begin{aligned}
\max \quad & z = 4x_1 + x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\
& 3x_1 - x_2 + x_3 \leq 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

1. $(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2.$

    1. Compare to target function – require expression bigger than target function in each variable.

$\implies z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2.$

Duality by Example: IV

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

1. $(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2.$
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\max \quad & z = 4x_1 + x_2 + 3x_3 \\
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& 3x_1 - x_2 + x_3 \leq 3 \\
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\end{aligned}
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1. $(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2.$

$$
\begin{aligned}
4 &\leq y_1 + 3y_2 \\
1 &\leq 4y_1 - y_2 \\
3 &\leq y_2,
\end{aligned}
$$

1. Compare to target function – require expression bigger than target function in each variable.

$\implies z = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2.$

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1. Compare to target function – require expression bigger than target function in each variable.

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Duality by Example: IV

Primal LP:

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Dual LP: $\widehat{L}$

$$\begin{aligned} \min \quad & y_1 + 3y_2 \\ \text{s.t.} \quad & y_1 + 3y_2 \geq 4 \\ & 4y_1 - y_2 \geq 1 \\ & y_2 \geq 3 \\ & y_1, y_2 \geq 0. \end{aligned}$$

1. Best upper bound on η (max value of z) then solve the LP $\widehat{L}$.
2. $\widehat{L}$: Dual program to L.
3. opt. solution of $\widehat{L}$ is an upper bound on optimal solution for L.

Duality by Example: IV

Primal LP:

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

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$$\begin{array}{lrl} \max & z = 4x_1 + x_2 + 3x_3 & \\ \text{s.t.} & x_1 + 4x_2 & \leq 1 \\ & 3x_1 - x_2 + x_3 & \leq 3 \\ & x_1, x_2, x_3 \geq 0 & \end{array}$$

Dual LP: $\widehat{L}$

$$\begin{array}{ll} \min & y_1 + 3y_2 \\ \text{s.t.} & y_1 + 3y_2 \geq 4 \\ & 4y_1 - y_2 \geq 1 \\ & y_2 \geq 3 \\ & y_1, y_2 \geq 0. \end{array}$$

1. Best upper bound on η (max value of z) then solve the LP $\widehat{L}$.
2. $\widehat{L}$: Dual program to L.
3. opt. solution of $\widehat{L}$ is an upper bound on optimal solution for L.

Primal program/Dual program

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \ge 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

$$\begin{aligned} \min \sum_{i=1}^{m} & \; b_i y_i \\ \text{s.t.} \sum_{i=1}^{m} & a_{ij} y_i \ge c_j, \\ & \qquad \text{for } j = 1, \ldots, n, \\ & y_i \ge 0, \\ & \qquad \text{for } i = 1, \ldots, m. \end{aligned}$$

Primal program/Dual program

Dual variables \ Primal variables	$x_1 \geqq 0$	$x_2 \geqq 0$	$x_3 \geqq 0$	$\cdots$	$x_n \geqq 0$	Primal relation	Min v
$y_1 \geqq 0$	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leqq$	b_1
$y_2 \geqq 0$	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leqq$	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$y_m \geqq 0$	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leqq$	b_m
Dual Relation	$\geqq$	$\geqq$	$\geqq$		$\geqq$		
Max z	c_1	c_2	c_3	$\cdots$	c_n		

$$\begin{aligned} \max \quad & c^T x \\ \text{s. t.} \quad & Ax \leq b. \\ & x \geq 0. \end{aligned}$$

$$\begin{aligned} \min \quad & y^T b \\ \text{s. t.} \quad & y^T A \geq c^T. \\ & y \geq 0. \end{aligned}$$

Primal program/Dual program

What happens when you take the dual of the dual?

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$
$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j = 1, \ldots, n,$$
$$y_i \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Primal program / Dual program in standard form

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$

$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$$\max \sum_{i=1}^{m} (-b_i) y_i$$

$$\text{s.t.} \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j, \quad \text{for } j = 1, \ldots, n,$$

$$y_i \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Dual program in standard form

Dual of a dual program

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} (-b_i) y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j, \\ & \qquad \text{for } j = 1, \ldots, n, \\ & y_i \geq 0, \\ & \qquad \text{for } i = 1, \ldots, m. \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} -c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i, \\ & \qquad \text{for } i = 1, \ldots, m, \\ & x_j \geq 0, \\ & \qquad \text{for } j = 1, \ldots, n. \end{aligned}$$

Dual of dual program

Dual of a dual program written in standard form

$$\min \sum_{j=1}^{n} -c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i, \quad \text{for } i = 1, \ldots, m,$$
$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$
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$\implies$ Dual of the dual LP is the primal LP!

Dual of dual program

Dual of a dual program written in standard form

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$$\text{s.t.} \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i, \quad \text{for } i = 1, \ldots, m,$$
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$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$
$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$\Longrightarrow$ Dual of the dual LP is the primal LP!

Result

Proved the following:

Lemma

Let L be an LP, and let L' be its dual. Let L'' be the dual to L'. Then L and L'' are the same LP.

20.2.2: The Weak Duality Theorem

Weak duality theorem

Theorem

If $(x_1, x_2, \ldots, x_n)$ *is feasible for the primal* LP *and* $(y_1, y_2, \ldots, y_m)$ *is feasible for the dual* LP, *then*

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

Namely, all the feasible solutions of the dual bound all the feasible solutions of the primal.

Weak duality theorem – proof

Proof.

By substitution from the dual form, and since the two solutions are feasible, we know that

$$\sum_j c_j x_j \leq \sum_j \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \leq \sum_i \left(\sum_j a_{ij} x_j \right) y_i \leq \sum_i b_i y_i \ .$$

□

1. y being dual feasible implies $c^T \leq y^T A$
2. x being primal feasible implies $Ax \leq b$
3. $\Rightarrow c^T x \leq (y^T A)x \leq y^T(Ax) \leq y^T b$

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□

1. y being dual feasible implies $c^T \leq y^T A$
2. x being primal feasible implies $Ax \leq b$
3. $\Rightarrow c^T x \leq (y^T A)x \leq y^T (Ax) \leq y^T b$

Weak duality is weak...

1. If apply the weak duality theorem on the dual program,
2. $$\implies \sum_{i=1}^{m} (-b_i) y_i \leq \sum_{j=1}^{n} -c_j x_j,$$
3. which is the original inequality in the weak duality theorem.
4. Weak duality theorem does not imply the strong duality theorem which will be discussed next.

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4. Weak duality theorem does not imply the strong duality theorem which will be discussed next.

20.3: The strong duality theorem

The strong duality theorem

Theorem (Strong duality theorem.)

If the primal LP problem has an optimal solution $x^* = (x_1^*, \ldots, x_n^*)$ *then the dual also has an optimal solution,* $y^* = (y_1^*, \ldots, y_m^*)$, *such that*

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

Proof is tedious and omitted.

20.4: Some duality examples

20.4.1: Maximum matching in Bipartite graph

Max matching in bipartite graph as LP

Input: $\mathsf{G} = (L \cup R, \mathsf{E})$.

$$
\begin{aligned}
\max \quad & \sum_{uv \in \mathsf{E}} x_{uv} & \\
s.t. \quad & \sum_{uv \in \mathsf{E}} x_{uv} \le 1 & \forall v \in \mathsf{G}. \\
& x_{uv} \ge 0 & \forall uv \in \mathsf{E}
\end{aligned}
$$

Max matching in bipartite graph as LP (Copy)

Input: $\mathbf{G} = (L \cup R, \mathbf{E})$.

$$
\begin{array}{lll}
\max & \displaystyle\sum_{uv \in \mathbf{E}} x_{uv} & \\
s.t. & \displaystyle\sum_{uv \in \mathbf{E}} x_{uv} \leq 1 & \forall v \in \mathbf{G}. \\
 & x_{uv} \geq 0 & \forall uv \in \mathbf{E}
\end{array}
$$

Max matching in bipartite graph as LP (Notes)

20.4.2: Shortest path

Shortest path

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: graph. $\mathbf{s}$: source , $\mathbf{t}$: target
2. $\forall (u, v) \in \mathbf{E}$: weight $\omega(u, v)$ on edge.
3. Q: Comp. shortest $\mathbf{s}$-$\mathbf{t}$ path.
4. No edges into $\mathbf{s}$/out of $\mathbf{t}$.
5. d_x: var=dist. $\mathbf{s}$ to x, $\forall x \in \mathbf{V}$.
6. $\forall (u, v) \in \mathbf{E}$: $d_u + \omega(u, v) \geq d_v$.
7. Also $d_{\mathbf{s}} = 0$.
8. Trivial solution: all variables 0.
9. Target: find assignment max $d_{\mathbf{t}}$.
10. LP to solve this!

Shortest path

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: graph. $\mathbf{s}$: source , $\mathbf{t}$: target
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8. Trivial solution: all variables 0.
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Shortest path

$$\begin{aligned} \max \quad & d_{\mathsf{t}} \\ \text{s.t.} \quad & d_{\mathsf{s}} \leq 0 \\ & d_u + \omega(u,v) \geq d_v \\ & \qquad\qquad \forall (u,v) \in \mathsf{E}, \\ & d_x \geq 0 \quad \forall x \in \mathsf{V}. \end{aligned}$$

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10. LP to solve this!

Shortest path

$$\begin{aligned} \max \quad & d_{\mathsf{t}} \\ \text{s.t.} \quad & d_{\mathsf{s}} \leq 0 \\ & d_u + \omega(u,v) \geq d_v \\ & \qquad\qquad \forall (u,v) \in \mathsf{E}, \\ & d_x \geq 0 \quad \forall x \in \mathsf{V}. \end{aligned}$$

Equivalently:

$$\begin{aligned} \max \quad & d_{\mathsf{t}} \\ \text{s.t.} \quad & d_{\mathsf{s}} \leq 0 \\ & d_v - d_u \leq \omega(u,v) \\ & \qquad\qquad \forall (u,v) \in \mathsf{E}, \\ & d_x \geq 0 \qquad \forall x \in \mathsf{V}. \end{aligned}$$

1. $\mathsf{G} = (\mathsf{V}, \mathsf{E})$: graph. **s**: source , **t**: target
2. $\forall (u,v) \in \mathsf{E}$: weight $\omega(u,v)$ on edge.
3. Q: Comp. shortest **s**-**t** path.
4. No edges into **s**/out of **t**.
5. d_x: var=dist. **s** to x, $\forall x \in \mathsf{V}$.
6. $\forall (u,v) \in \mathsf{E}$: $d_u + \omega(u,v) \geq d_v$.
7. Also $d_{\mathsf{s}} = 0$.
8. Trivial solution: all variables 0.
9. Target: find assignment max d_{t}.
10. LP to solve this!

The dual

$$\begin{aligned}
\min \quad & \sum_{(u,v)\in \mathsf{E}} y_{uv}\,\omega(u,v) \\
\text{s.t.} \quad & y_{\mathsf{s}} - \sum_{(\mathsf{s},u)\in \mathsf{E}} y_{\mathsf{s}u} \geq 0 & (*)\\
& \sum_{(u,x)\in \mathsf{E}} y_{ux} - \sum_{(x,v)\in \mathsf{E}} y_{xv} \geq 0 \\
& \qquad \forall x \in \mathsf{V} \setminus \{\mathsf{s},\mathsf{t}\} & (**)\\
& \sum_{(u,\mathsf{t})\in \mathsf{E}} y_{u\mathsf{t}} \geq 1 & (***)\\
& y_{uv} \geq 0, \quad \forall (u,v) \in \mathsf{E}, \\
& y_{\mathsf{s}} \geq 0.
\end{aligned}$$

$$\begin{aligned}
\max \quad & d_{\mathsf{t}} \\
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The dual – details

1. y_{uv}: dual variable for the edge (u, v).
2. y_s: dual variable for $d_s \leq 0$
3. Think about the y_{uv} as a flow on the edge y_{uv}.
4. Assume that weights are positive.
5. LP is min cost flow of sending 1 unit flow from source s to t.
6. Indeed... (**) can be assumed to be hold with equality in the optimal solution...
7. conservation of flow.
8. Equation (***) implies that one unit of flow arrives to the sink t.
9. (*) implies that at least y_s units of flow leaves the source.
10. Remaining of LP implies that $y_s \geq 1$.

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Integrality

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Set cover...

Details in notes...

Set cover LP:

$$
\begin{aligned}
\min \quad & \sum_{F_j \in \mathfrak{F}} x_j & \\
\text{s.t.} \quad & \sum_{\substack{F_j \in \mathfrak{F}, \\ u_i \in F_j}} x_j \geq 1 & \forall u_i \in \mathsf{S}, \\
& x_j \geq 0 & \forall F_j \in \mathfrak{F}.
\end{aligned}
$$

Set cover dual is a packing LP...

Details in notes...

$$
\begin{aligned}
\max \quad & \sum_{u_i \in \mathsf{S}} y_i & \\
\text{s.t.} \quad & \sum_{u_i \in F_j} y_i \le 1 & \forall F_j \in \mathfrak{F}, \\
& y_i \ge 0 & \forall u_i \in \mathsf{S}.
\end{aligned}
$$

Network flow

$$
\begin{aligned}
\max \quad & \sum_{(\mathsf{s},v)\in\mathsf{E}} x_{\mathsf{s}\to v} & \\
& x_{u\to v} \le \mathsf{c}(u\to v) & \forall (u,v)\in\mathsf{E}\\
& \sum_{(u,v)\in\mathsf{E}} x_{u\to v} - \sum_{(v,w)\in\mathsf{E}} x_{v\to w} \le 0 & \forall v\in\mathsf{V}\setminus\{\mathsf{s},\mathsf{t}\}\\
& -\sum_{(u,v)\in\mathsf{E}} x_{u\to v} + \sum_{(v,w)\in\mathsf{E}} x_{v\to w} \le 0 & \forall v\in\mathsf{V}\setminus\{\mathsf{s},\mathsf{t}\}\\
& 0 \le x_{u\to v} & \forall (u,v)\in\mathsf{E}.
\end{aligned}
$$

Dual of network flow...

$$\min \sum_{(u,v)\in \mathsf{E}} \mathsf{c}(u \to v)\, y_{u\to v}$$

$$d_u - d_v \leq y_{u\to v} \qquad \forall (u,v) \in \mathsf{E}$$

$$y_{u\to v} \geq 0 \qquad \forall (u,v) \in \mathsf{E}$$

$$d_{\mathsf{s}} = 1, \qquad d_{\mathsf{t}} = 0.$$

Under right interpretation: shortest path (see notes).

Duality and min-cut max-flow

Details in class notes

Lemma

The Min-Cut Max-Flow Theorem follows from the strong duality Theorem for Linear Programming.

Notes

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