NEW CS 473: Theory II, Fall 2015

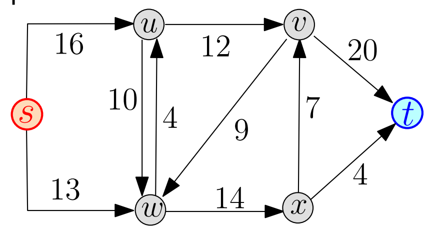
Network flow, duality and Linear Programming

Lecture 20

November 5, 2015

Network flow

1. Transfer as much “merchandise” as possible from one point to another.
2. Wireless network, transfer a large file from s to t.
3. Limited capacities.



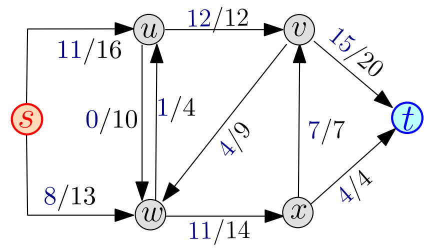
Network: Definition

1. Given a network with capacities on each connection.
2. Q: How much “flow” can transfer from source s to a sink t?
3. The flow is **splitable**.
4. Network examples: water pipes moving water. Electricity network.
5. Internet is packet base, so not quite splitable.

Definition

⋆ $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: a **directed** graph.
⋆ $\forall (u, v) \in \mathbf{E}(\mathbf{G})$: **capacity** $c(u, v) \geq 0$,
⋆ $(u, v) \notin G \implies c(u, v) = 0$.
⋆ s: **source** vertex, t: target **sink** vertex.
⋆ $\mathbf{G}$, s, t and $c(\cdot)$: form **flow network** or **network**.

Network Example



1. All flow from the source ends up in the sink.
2. Flow on edge: non-negative quantity $\leq$ capacity of edge.

Flow definition

Definition (flow)

flow in network is a function $f(\cdot, \cdot) : E(G) \rightarrow \mathbb{R}$:

(A) **Bounded by capacity**:
$\forall (u, v) \in E \quad f(u, v) \leq c(u, v)$.

(B) **Anti symmetry**:
$\forall u, v \quad f(u, v) = -f(v, u)$.

(C) Two special vertices: (i) the **source** s and the **sink** t.

(D) **Conservation of flow** (Kirchhoff's Current Law):
$\forall u \in V \setminus \{s, t\} \quad \sum_{v} f(u, v) = 0$.

flow/**value** of f: $|f| = \sum_{v \in V} f(s, v)$.

Problem: Max Flow

1. Flow on edge can be negative (i.e., positive flow on edge in other direction).

Problem (Maximum flow)

Given a network **G** *find the* **maximum flow** *in* **G**. *Namely, compute a legal flow* f *such that* $|f|$ *is maximized.*

Network flow via linear programming

Input: $G = (V, E)$ with source **s** and sink **t**, and capacities $c(\cdot)$ on the edges. Compute max flow in **G**.

$$
\begin{array}{ll}
\forall (u, v) \in E & 0 \leq x_{u\rightarrow v} \\
 & x_{u\rightarrow v} \leq c(u \rightarrow v) \\
\forall v \in V \setminus \{s, t\} & \sum_{(u,v)\in E} x_{u\rightarrow v} - \sum_{(v,w)\in E} x_{v\rightarrow w} \leq 0 \\
 & \sum_{(u,v)\in E} x_{u\rightarrow v} - \sum_{(v,w)\in E} x_{v\rightarrow w} \geq 0 \\
\text{maximizing} & \sum_{(s,u)\in E} x_{s\rightarrow u}
\end{array}
$$

Min cost flow

Input:

$G = (V, E)$: directed graph.
s: source.
t: sink
$c(\cdot)$: capacities on edges,
ϕ: Desired amount (**value**) of flow.
$\kappa(\cdot)$: Cost on the edges.

Definition - cost of flow

cost of flow **f**: $\text{cost}(f) = \sum_{e\in E} \kappa(e) * f(e)$.

Min cost flow problem

Min-cost flow

minimum-cost s-t flow problem: compute the flow $\mathbf{f}$ of min cost that has value ϕ.

min-cost circulation problem

Instead of ϕ we have lower-bound $\ell(\cdot)$ on edges.
(All flow that enters must leave.)

Claim

If we can solve min-cost circulation $\implies$ can solve min-cost flow.

Duality...

1. Every linear program L has a **dual linear program** L'.
2. Solving the dual problem is essentially equivalent to solving the **primal linear program** original LP.
3. Lets look an example..

Duality by Example

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. η: maximal possible value of target function.
2. Any feasible solution $\Rightarrow$ a lower bound on η.
3. In above: $x_1 = 1, x_2 = x_3 = 0$ is feasible, and implies $z = 4$ and thus $\eta \geq 4$.
4. $x_1 = x_2 = 0$, $x_3 = 3$ is feasible $\implies \eta \geq z = 9$.
5. How close this solution is to opt? (i.e., η)
6. If very close to optimal – might be good enough. Maybe stop?

Duality by Example: II

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. Add the first inequality (multiplied by 2) to the second inequality (multiplied by 3):

$$\begin{array}{r} 2(\;x_1 + 4x_2 \qquad) \leq 2(1) \\ +3(3x_1 - x_2 + x_3) \leq 3(3). \end{array}$$

2. The resulting inequality is

$$11x_1 + 5x_2 + 3x_3 \leq 11. \qquad (1)$$

Duality by Example: II

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \ \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. got $11x_1 + 5x_2 + 3x_3 \leq 11$.
2. inequality must hold for any feasible solution of L.
3. Objective: $z = 4x_1 + x_2 + 3x_3$ and x_1,x_2 and x_3 are all non-negative.
4. Inequality above has larger coefficients than objective (for corresponding variables)
5. For any feasible solution: $z = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$,

Duality by Example: III

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \ \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. For any feasible solution: $z = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq 11$,
2. Opt solution is LP L is somewhere between 9 and 11.
3. Multiply first inequality by y_1, second inequality by y_2 and add them up:

$$\begin{array}{ccccccc} y_1(x_1 & + & 4x_2 & & &) \leq & y_1(1) \\ + \; y_2(3x_1 & - & x_2 & + & x_3 &) \leq & y_2(3) \\ \hline (y_1 + 3y_2)x_1 & + & (4y_1 - y_2)x_2 & + & y_2 x_3 & \leq & y_1 + 3y_2. \end{array}$$

Duality by Example: IV

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \ \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

1. $(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2$.

$$\begin{array}{rcl} 4 & \leq & y_1 + 3y_2 \\ 1 & \leq & 4y_1 - y_2 \\ 3 & \leq & y_2, \end{array}$$

1. Compare to target function – require expression bigger than target function in each variable.

$\implies z = 4x_1 + x_2 + 3x_3 \leq$
$(y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2 x_3 \leq y_1 + 3y_2$.

Duality by Example: IV

Primal LP:

$$\begin{array}{rl} \max & z = 4x_1 + x_2 + 3x_3 \\ \text{s.t.} & x_1 + 4x_2 \ \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Dual LP: $\widehat{L}$

$$\begin{array}{rl} \min & y_1 + 3y_2 \\ \text{s.t.} & y_1 + 3y_2 \geq 4 \\ & 4y_1 - y_2 \geq 1 \\ & y_2 \geq 3 \\ & y_1, y_2 \geq 0. \end{array}$$

1. Best upper bound on η (max value of z) then solve the LP $\widehat{L}$.
2. $\widehat{L}$: Dual program to L.
3. opt. solution of $\widehat{L}$ is an upper bound on optimal solution for L.

Primal program/Dual program

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$
$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j = 1, \ldots, n,$$
$$y_i \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Primal program/Dual program

Dual variables \ Primal variables	$x_1 \geqq 0$	$x_2 \geqq 0$	$x_3 \geqq 0$	$\cdots$	$x_n \geqq 0$	Primal relation	Min v
$y_1 \geqq 0$	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leqq$	b_1
$y_2 \geqq 0$	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leqq$	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$y_m \geqq 0$	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leqq$	b_m
Dual Relation	$\geqq$	$\geqq$	$\geqq$		$\geqq$		
Max z	c_1	c_2	c_3	$\cdots$	c_n		

$$\max \; c^T x \quad \text{s. t.} \; Ax \leq b. \quad x \geq 0.$$

$$\min \; y^T b \quad \text{s. t.} \; y^T A \geq c^T. \quad y \geq 0.$$

Primal program/Dual program

What happens when you take the dual of the dual?

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$
$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j = 1, \ldots, n,$$
$$y_i \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Primal program / Dual program in standard form

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i = 1, \ldots, m,$$
$$x_j \geq 0, \quad \text{for } j = 1, \ldots, n.$$

$$\max \sum_{i=1}^{m} (-b_i) y_i$$
$$\text{s.t.} \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j, \quad \text{for } j = 1, \ldots, n,$$
$$y_i \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Dual program in standard form

Dual of a dual program

$$\max \sum_{i=1}^{m} (-b_i) y_i$$
$$\text{s.t.} \sum_{i=1}^{m} (-a_{ij}) y_i \leq -c_j,$$
$$\text{for } j = 1, \ldots, n,$$
$$y_i \geq 0,$$
$$\text{for } i = 1, \ldots, m.$$

$$\min \sum_{j=1}^{n} -c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i,$$
$$\text{for } i = 1, \ldots, m,$$
$$x_j \geq 0,$$
$$\text{for } j = 1, \ldots, n.$$

Dual of dual program

Dual of a dual program written in standard form

$$\min \sum_{j=1}^{n} -c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} (-a_{ij}) x_j \geq -b_i,$$
$$\text{for } i = 1, \ldots, m,$$
$$x_j \geq 0,$$
$$\text{for } j = 1, \ldots, n.$$

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i,$$
$$\text{for } i = 1, \ldots, m,$$
$$x_j \geq 0,$$
$$\text{for } j = 1, \ldots, n.$$

$\Longrightarrow$ Dual of the dual LP is the primal LP!

Result

Proved the following:

Lemma

Let L be an LP, and let L' be its dual. Let L'' be the dual to L'. Then L and L'' are the same LP.

Weak duality theorem

Theorem

If $(x_1, x_2, \ldots, x_n)$ is feasible for the primal LP and $(y_1, y_2, \ldots, y_m)$ is feasible for the dual LP, then

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

Namely, all the feasible solutions of the dual bound all the feasible solutions of the primal.

Weak duality theorem – proof

Proof.
By substitution from the dual form, and since the two solutions are feasible, we know that

$$\sum_j c_j x_j \le \sum_j \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \le \sum_i \left(\sum_j a_{ij} x_j \right) y_i \le \sum_i b_i y_i \,.$$

□

1. y being dual feasible implies $c^T \le y^T A$
2. x being primal feasible implies $Ax \le b$
3. $\Rightarrow c^T x \le (y^T A)x \le y^T(Ax) \le y^T b$

Weak duality is weak...

1. If apply the weak duality theorem on the dual program,
2. $\implies \sum_{i=1}^{m} (-b_i) y_i \le \sum_{j=1}^{n} -c_j x_j,$
3. which is the original inequality in the weak duality theorem.
4. Weak duality theorem does not imply the strong duality theorem which will be discussed next.

The strong duality theorem

Theorem (Strong duality theorem.)
If the primal LP problem has an optimal solution $x^* = (x_1^*, \ldots, x_n^*)$ then the dual also has an optimal solution, $y^* = (y_1^*, \ldots, y_m^*)$, such that

$$\sum_j c_j x_j^* = \sum_i b_i y_i^* .$$

Proof is tedious and omitted.

Max matching in bipartite graph as LP

Input: $\mathbf{G} = (L \cup R, \mathbf{E})$.

$$\begin{aligned}
\max \quad & \sum_{uv \in \mathbf{E}} x_{uv} & \\
s.t. \quad & \sum_{uv \in \mathbf{E}} x_{uv} \le 1 & \forall v \in \mathbf{G}. \\
& x_{uv} \ge 0 & \forall uv \in \mathbf{E}
\end{aligned}$$

Max matching in bipartite graph as LP (Copy)

Input: $\mathbf{G} = (L \cup R, \mathbf{E})$.

$$\begin{array}{lll} \max & \sum_{uv \in \mathbf{E}} x_{uv} & \\ s.t. & \sum_{uv \in \mathbf{E}} x_{uv} \leq 1 & \forall v \in \mathbf{G}. \\ & x_{uv} \geq 0 & \forall uv \in \mathbf{E} \end{array}$$

Max matching in bipartite graph as LP (Notes)

Shortest path

$$\begin{array}{ll} \max & d_t \\ \text{s.t.} & d_s \leq 0 \\ & d_u + \omega(u,v) \geq d_v \\ & \qquad \forall (u,v) \in \mathbf{E}, \\ & d_x \geq 0 \quad \forall x \in \mathbf{V}. \end{array}$$

Equivalently:

$$\begin{array}{ll} \max & d_t \\ \text{s.t.} & d_s \leq 0 \\ & d_v - d_u \leq \omega(u,v) \\ & \qquad \forall (u,v) \in \mathbf{E}, \\ & d_x \geq 0 \quad \forall x \in \mathbf{V}. \end{array}$$

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: graph. $\mathbf{s}$: source , $\mathbf{t}$: target
2. $\forall (u,v) \in \mathbf{E}$: weight $\omega(u,v)$ on edge.
3. Q: Comp. shortest $\mathbf{s}$-$\mathbf{t}$ path.
4. No edges into $\mathbf{s}$/out of $\mathbf{t}$.
5. d_x: var=dist. $\mathbf{s}$ to x, $\forall x \in \mathbf{V}$.
6. $\forall (u,v) \in \mathbf{E}$: $d_u + \omega(u,v) \geq d_v$.
7. Also $d_s = 0$.
8. Trivial solution: all variables 0.
9. Target: find assignment max d_t.
10. LP to solve this!

The dual

$$\begin{array}{lll} \min & \sum_{(u,v) \in \mathbf{E}} y_{uv}\omega(u,v) & \\ \text{s.t.} & y_s - \sum_{(s,u) \in \mathbf{E}} y_{su} \geq 0 & (*) \\ & \sum_{(u,x) \in \mathbf{E}} y_{ux} - \sum_{(x,v) \in \mathbf{E}} y_{xv} \geq 0 & \\ & \qquad \forall x \in \mathbf{V} \setminus \{s,t\} & (**) \\ & \sum_{(u,t) \in \mathbf{E}} y_{ut} \geq 1 & (***) \\ & y_{uv} \geq 0, \quad \forall (u,v) \in \mathbf{E}, & \\ & y_s \geq 0. & \end{array}$$

$$\begin{array}{ll} \max & d_t \\ \text{s.t.} & d_s \leq 0 \\ & d_v - d_u \leq \omega(u,v) \\ & \qquad \forall (u,v) \in \mathbf{E}, \\ & d_x \geq 0 \quad \forall x \in \mathbf{V}. \end{array}$$

The dual – details

1. y_{uv}: dual variable for the edge (u, v).
2. y_s: dual variable for $d_s \leq 0$
3. Think about the y_{uv} as a flow on the edge y_{uv}.
4. Assume that weights are positive.
5. LP is min cost flow of sending 1 unit flow from source s to t.
6. Indeed... (**) can be assumed to be hold with equality in the optimal solution...
7. conservation of flow.
8. Equation (***) implies that one unit of flow arrives to the sink t.
9. (*) implies that at least y_s units of flow leaves the source.
10. Remaining of LP implies that $y_s \geq 1$.

Integrality

1. In the previous example there is always an optimal solution with integral values.
2. This is not an obvious statement.
3. This is not true in general.
4. If it were true we could solve **NPC** problems with LP.

Set cover...

Details in notes...

Set cover LP:

$$
\begin{aligned}
\min \quad & \sum_{F_j \in \mathfrak{F}} x_j & \\
\text{s.t.} \quad & \sum_{\substack{F_j \in \mathfrak{F},\\ u_i \in F_j}} x_j \geq 1 & \forall u_i \in S, \\
& x_j \geq 0 & \forall F_j \in \mathfrak{F}.
\end{aligned}
$$

Set cover dual is a packing LP...

Details in notes...

$$
\begin{aligned}
\max \quad & \sum_{u_i \in S} y_i & \\
\text{s.t.} \quad & \sum_{u_i \in F_j} y_i \leq 1 & \forall F_j \in \mathfrak{F}, \\
& y_i \geq 0 & \forall u_i \in S.
\end{aligned}
$$

Network flow

$$\begin{aligned}
\max \quad & \sum_{(s,v)\in E} x_{s\to v} & \\
& x_{u\to v} \leq c(u \to v) & \forall (u,v) \in E \\
& \sum_{(u,v)\in E} x_{u\to v} - \sum_{(v,w)\in E} x_{v\to w} \leq 0 & \forall v \in V \setminus \{s,t\} \\
& -\sum_{(u,v)\in E} x_{u\to v} + \sum_{(v,w)\in E} x_{v\to w} \leq 0 & \forall v \in V \setminus \{s,t\} \\
& 0 \leq x_{u\to v} & \forall (u,v) \in E.
\end{aligned}$$

Dual of network flow...

$$\begin{aligned}
\min \quad & \sum_{(u,v)\in E} c(u \to v)\, y_{u\to v} & \\
& d_u - d_v \leq y_{u\to v} & \forall (u,v) \in E \\
& y_{u\to v} \geq 0 & \forall (u,v) \in E \\
& d_s = 1, \qquad d_t = 0. &
\end{aligned}$$

Under right interpretation: shortest path (see notes).

Duality and min-cut max-flow

Details in class notes

Lemma

The Min-Cut Max-Flow Theorem follows from the strong duality Theorem for Linear Programming.