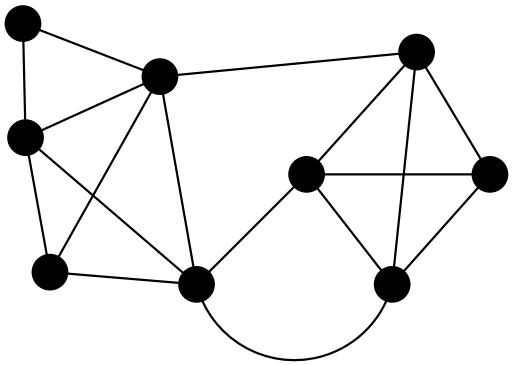
Randomized Algorithms III – Min Cut

Lecture 14

October 13, 2015

14.2: Min cut

Min cut



G $= (V, E)$: undirected graph, n vertices, m edges.

Interested in **cuts** in **G**.

Definition

cut in **G**: a partition of V: S and $V \setminus S$.

Edges of the cut:

$$(S, V \setminus S) = \left\{ uv \;\middle|\; u \in S, v \in V \setminus S, \text{ and } uv \in E \right\},$$

$|(S, V \setminus S)|$ is *size of the cut*

minimum cut / **mincut**: cut in graph with min size.

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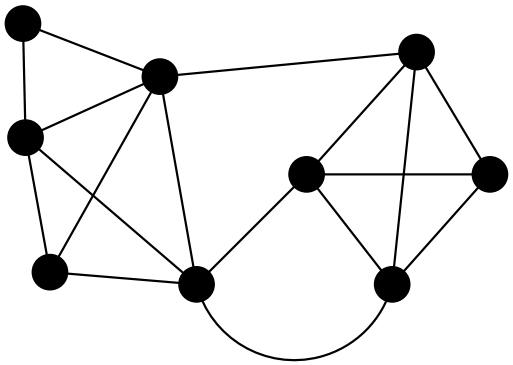

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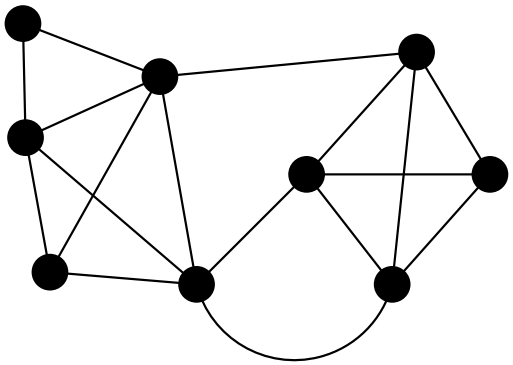

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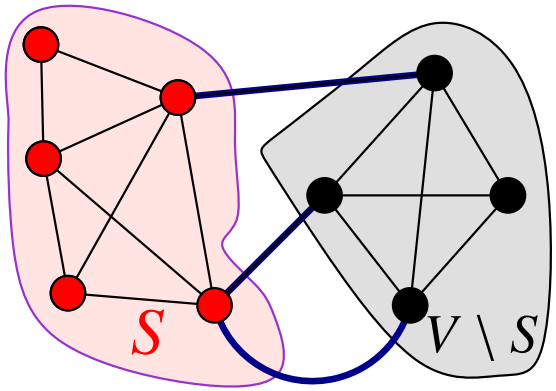

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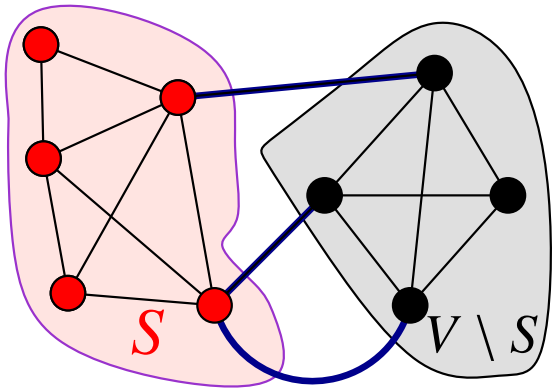

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Some definitions

1. **conditional probability** of X given Y is

$$\Pr\left[X = x \,|Y = y\right] = \frac{\Pr\left[(X=x)\cap(Y=y)\right]}{\Pr\left[Y=y\right]}.$$

$$\Pr\left[X = x \cap Y = y\right] = \Pr\left[X = x \,\middle|\, Y = y\right] \cdot \Pr[Y = y].$$

2. X, Y events are **independent**, if

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Some more probability

Lemma

$\mathcal{E}_1, \ldots, \mathcal{E}_n$: n events (not necessarily independent). Then,

$$\mathbf{Pr}\left[\cap_{i=1}^{n} \mathcal{E}_i\right] = \mathbf{Pr}\left[\mathcal{E}_1\right] * \mathbf{Pr}\left[\mathcal{E}_2 \,|\mathcal{E}_1\right] * \mathbf{Pr}\left[\mathcal{E}_3 \;\middle|\; \mathcal{E}_1 \cap \mathcal{E}_2\right] * \ldots$$
$$* \mathbf{Pr}\left[\mathcal{E}_n \;\middle|\; \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-1}\right].$$

Edge contraction...

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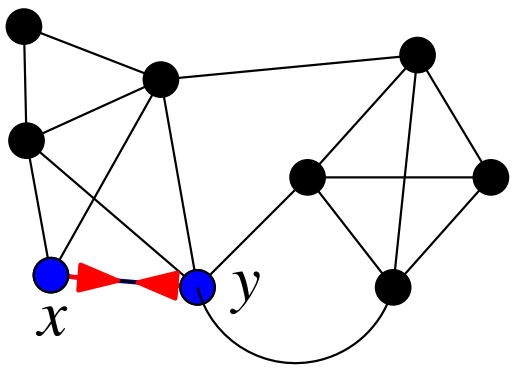

1. **edge contraction**: $e = xy$ in **G**.
2. ... merge x, y into a single vertex.
3. ...remove self loops.
4. ... parallel edges – **multi-graph**.
5. ... weights/ multiplicities on the edges.

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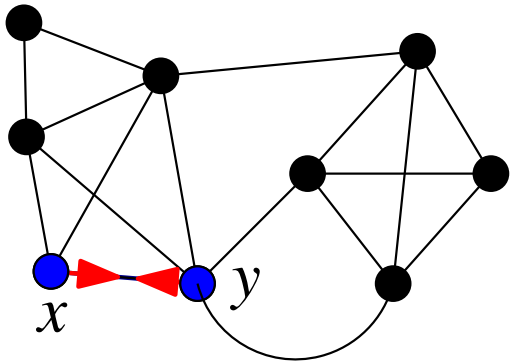

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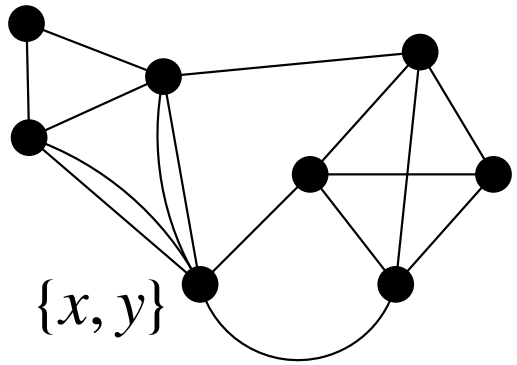

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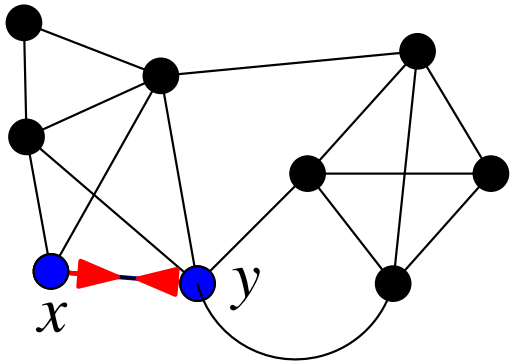

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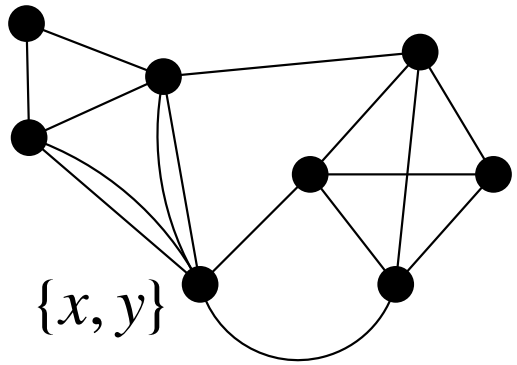

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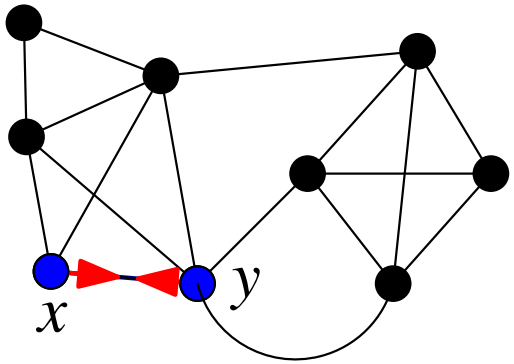

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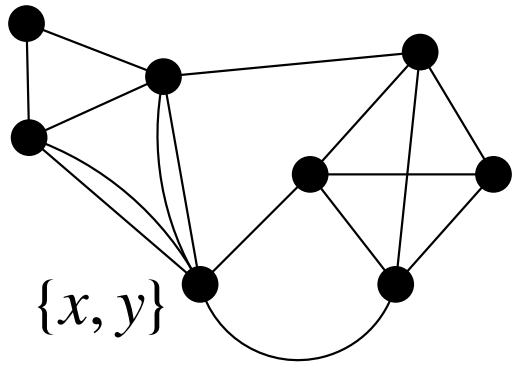

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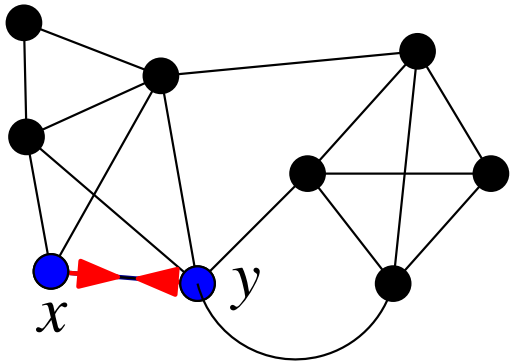

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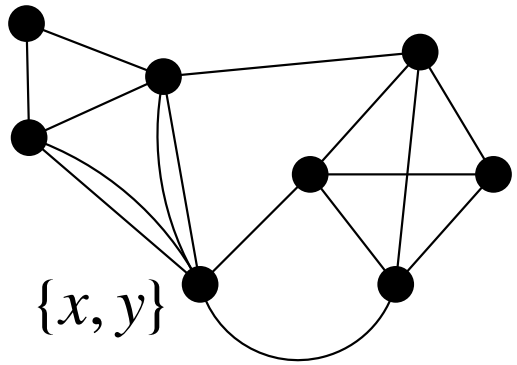

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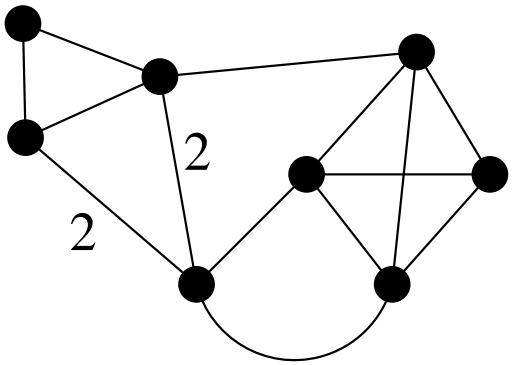

Edge contraction implemented in $O(n)$ time:

1. Graph represented using adjacency lists.
2. Merging the adjacency lists of the two vertices being contracted.
3. Fix adjacency list of vertices connected to x, y.
   [Use radix sort.]
4. Include edge weight in computing cut weight.

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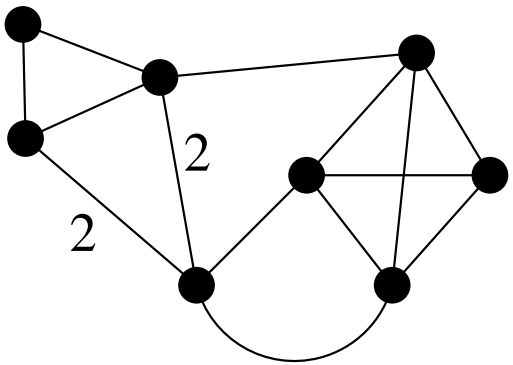

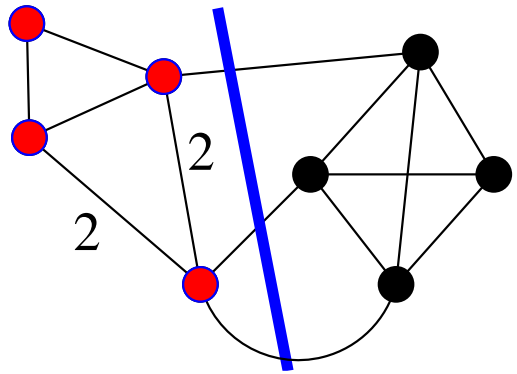

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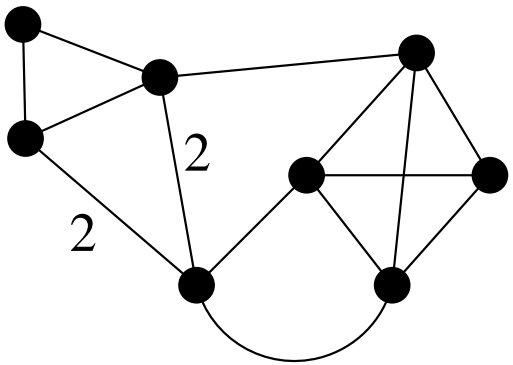

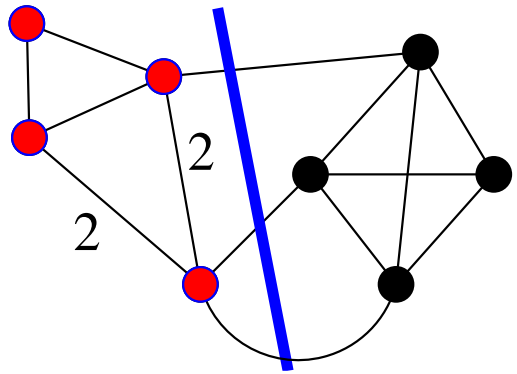

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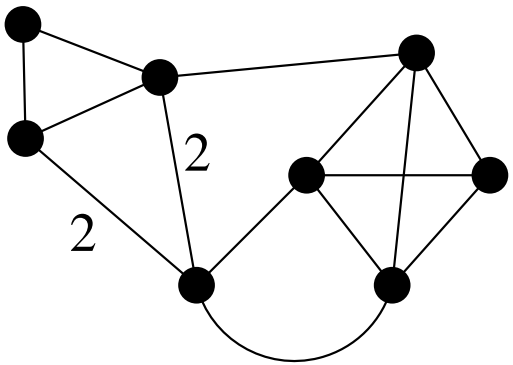

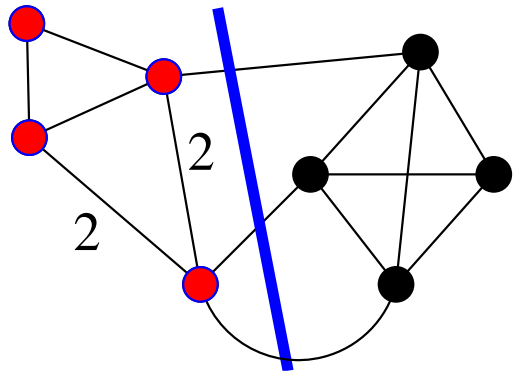

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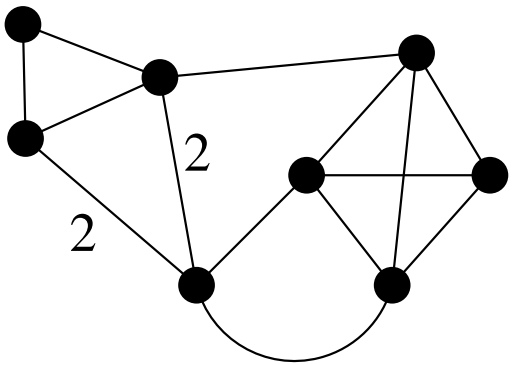

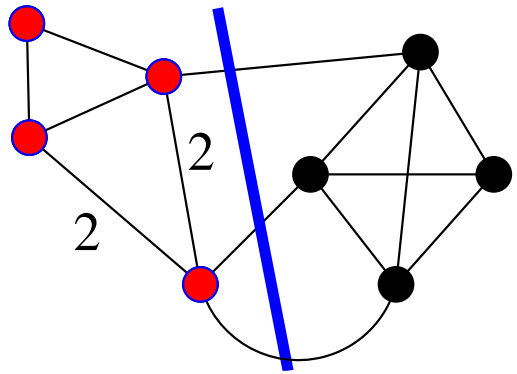

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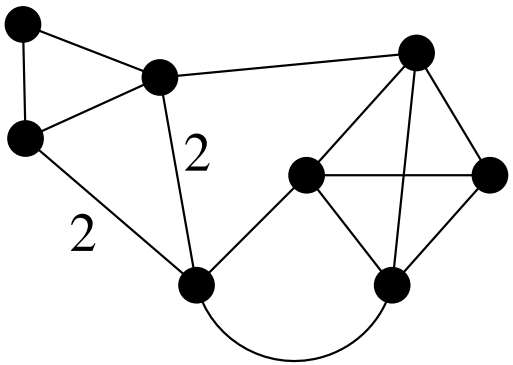

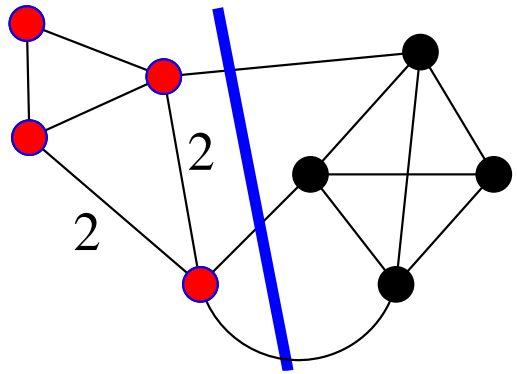

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2. *There* $\exists$ *cuts in* $\mathbf{G}$ *are not in* $\mathbf{G}/xy$.
3. *The cut* $S = \{x\}$ *is not in* $\mathbf{G}/xy$.
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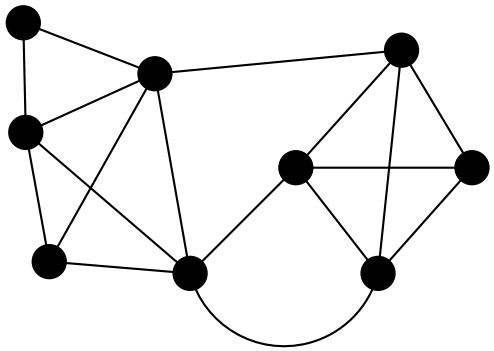

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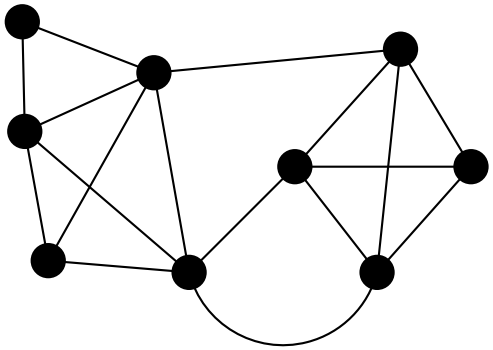

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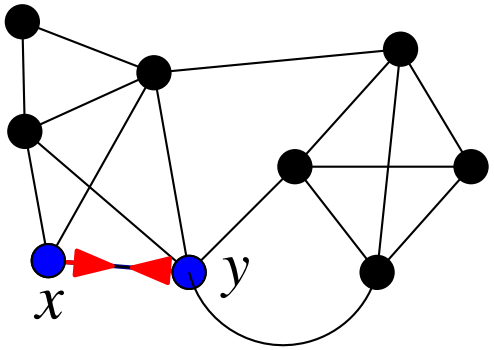

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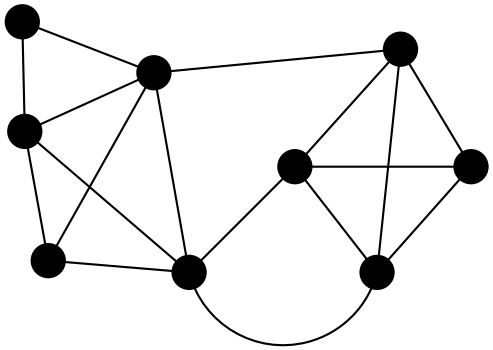

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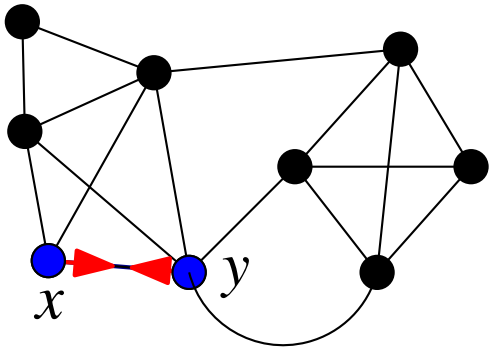

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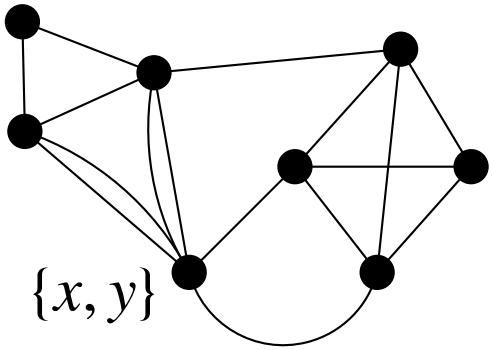

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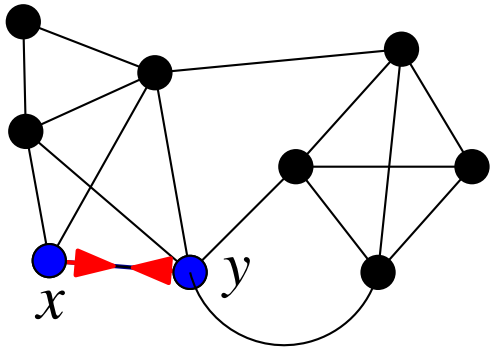

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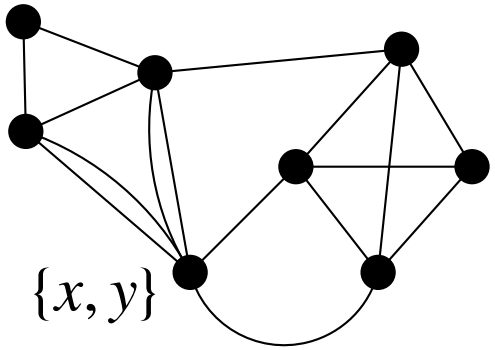

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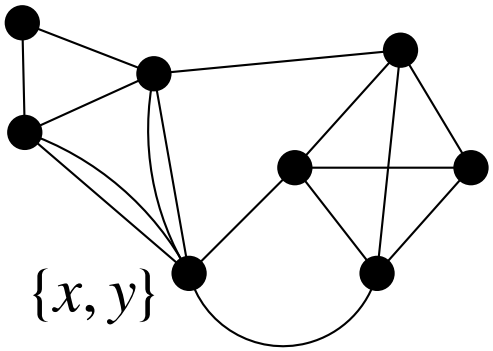

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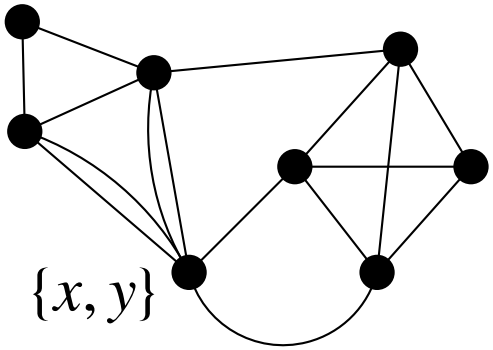
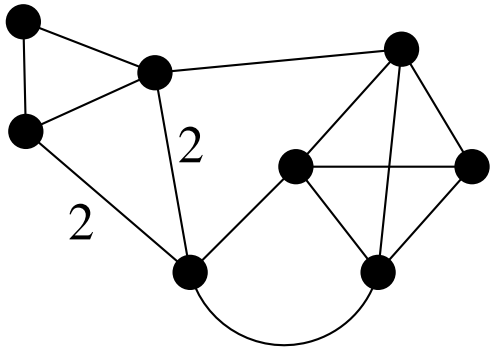


(4)



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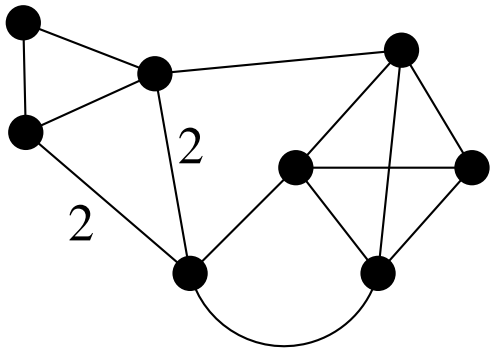

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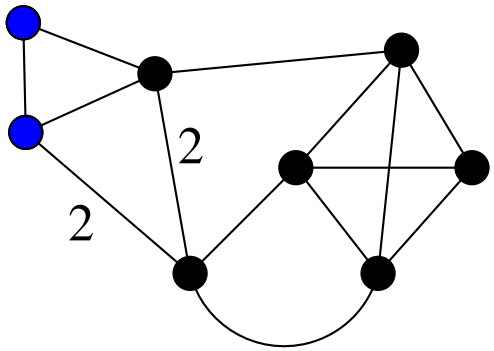

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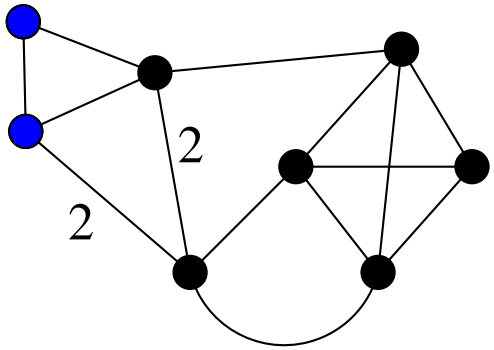

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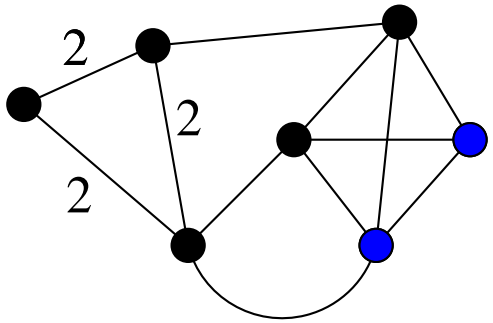

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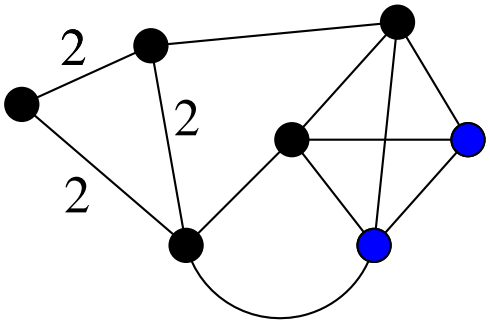

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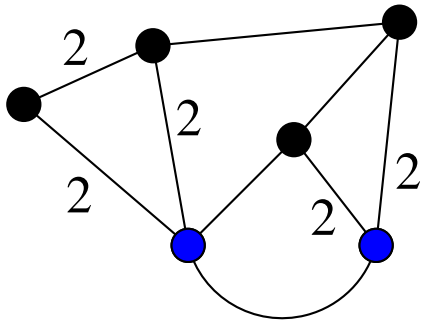

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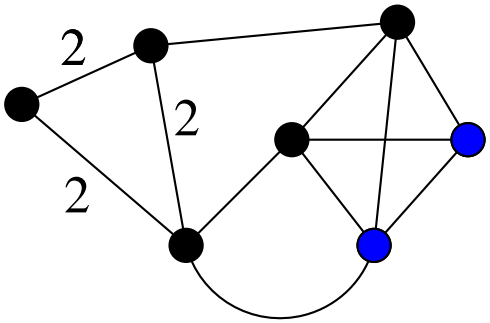

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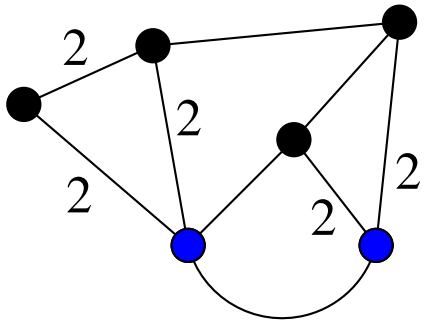

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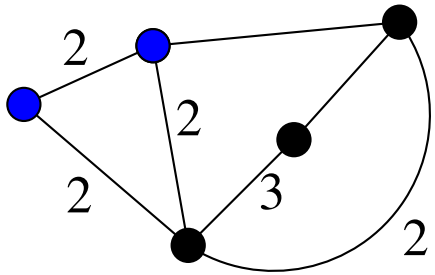

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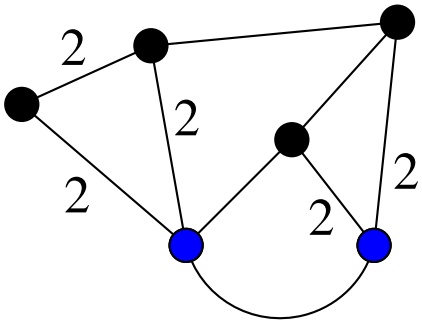

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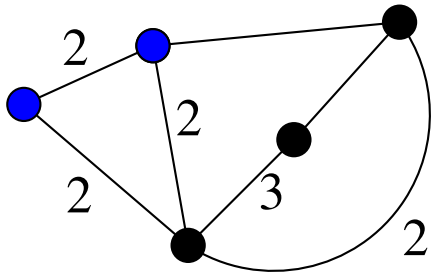

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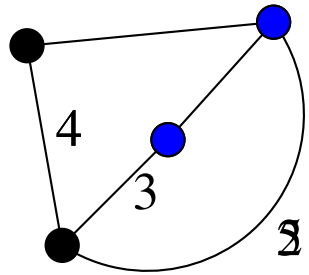

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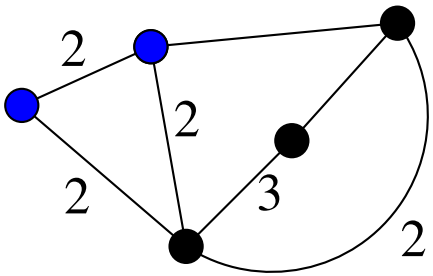

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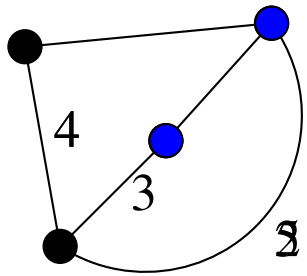

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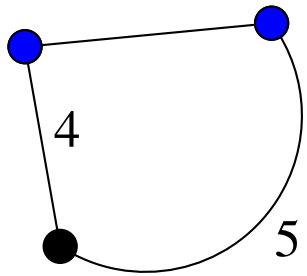

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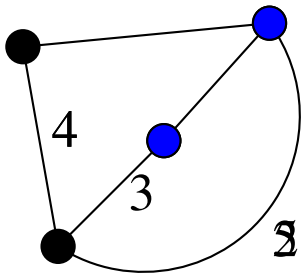

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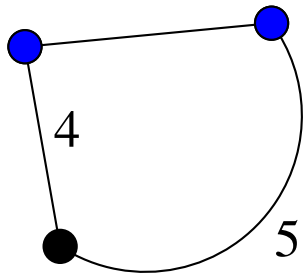

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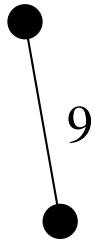

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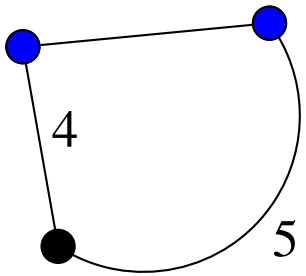

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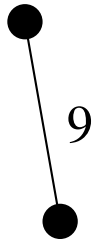

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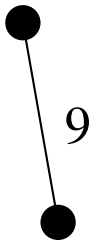

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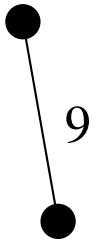

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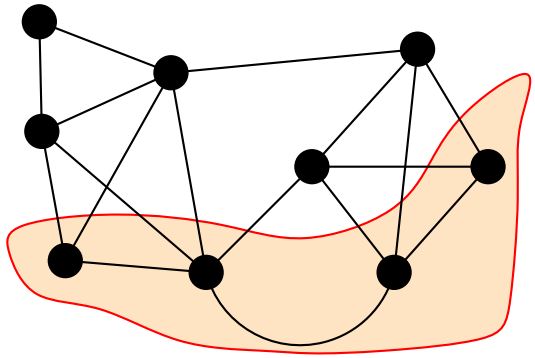

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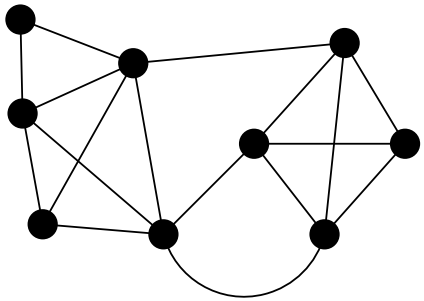
Contraction - all together now



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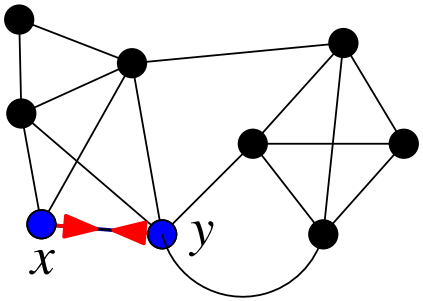

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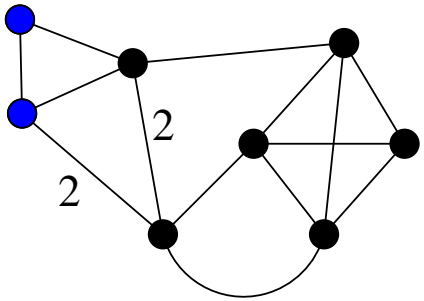

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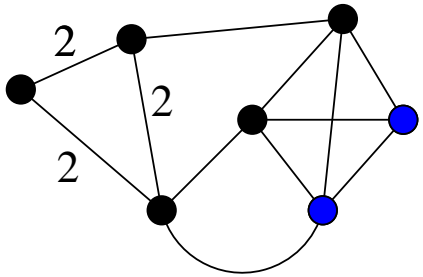

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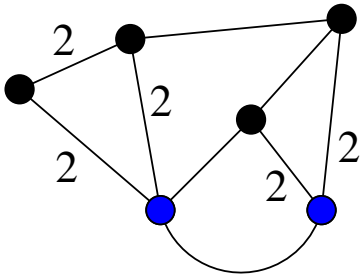

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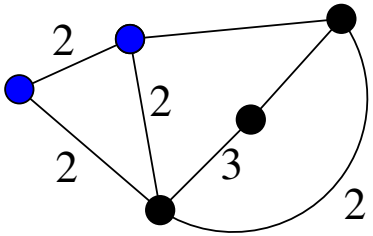

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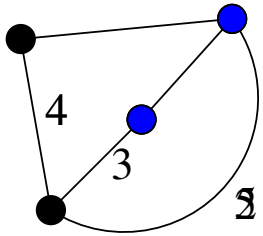

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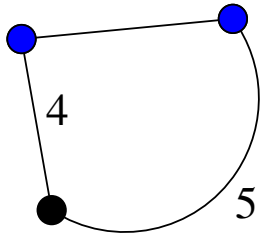

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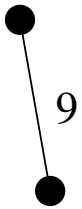

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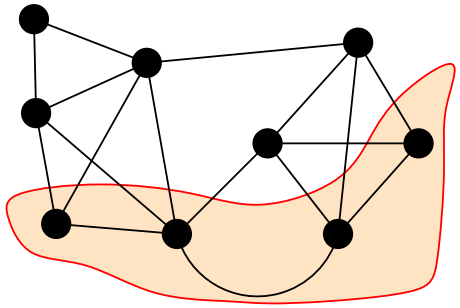
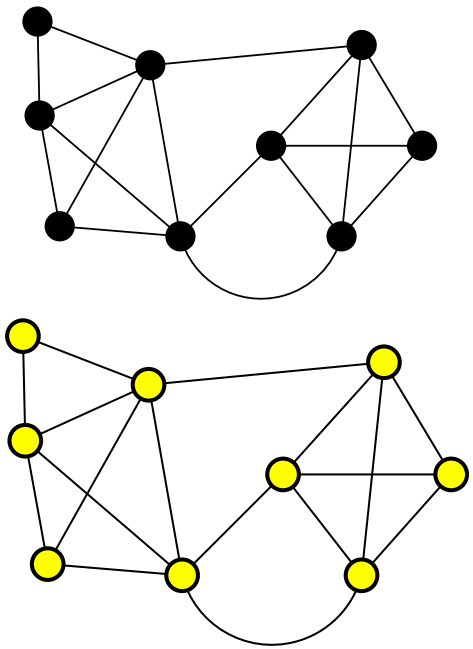


(j)

Another interpretation



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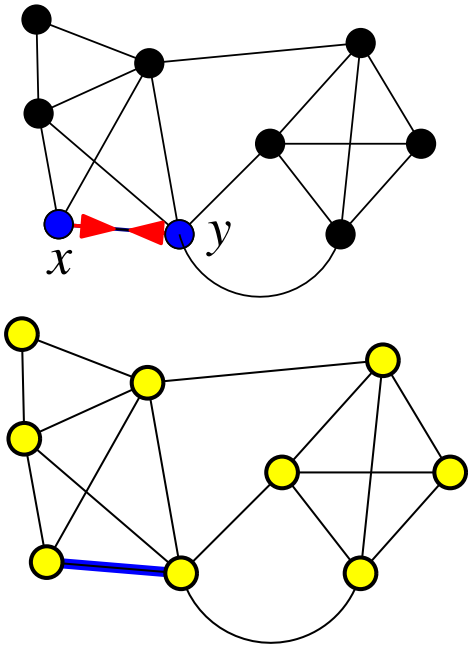

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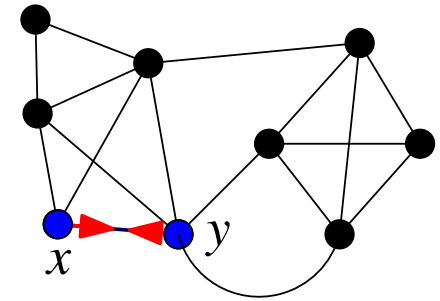

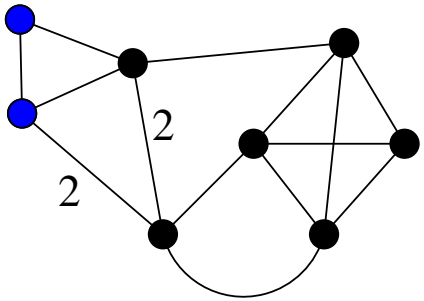

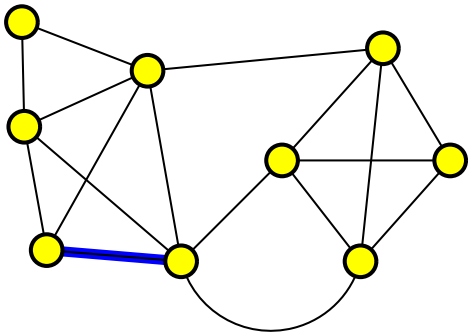

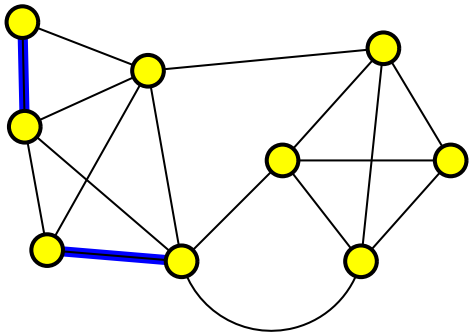

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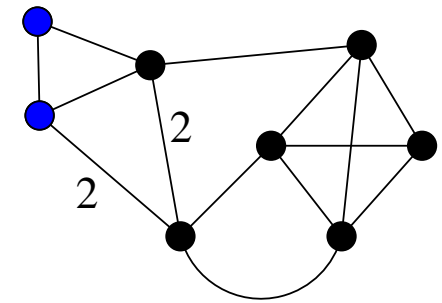

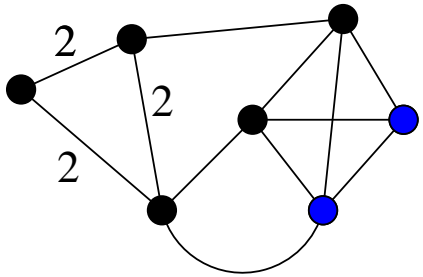

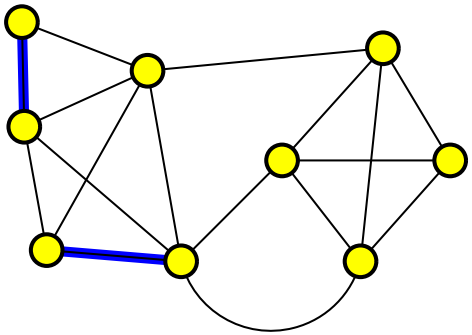

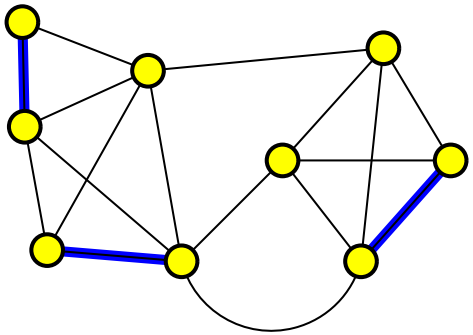

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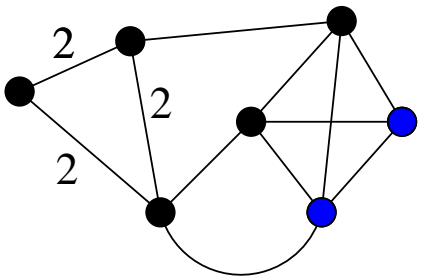

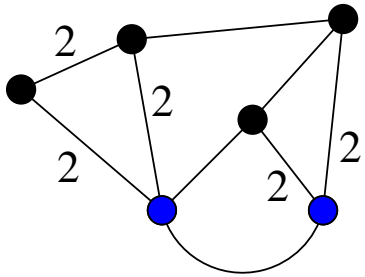

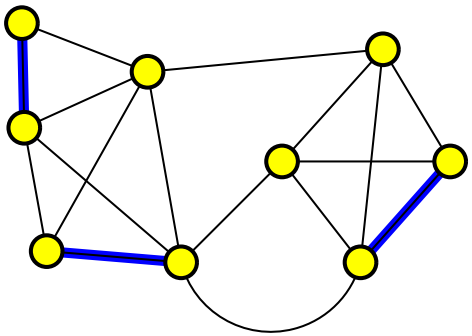

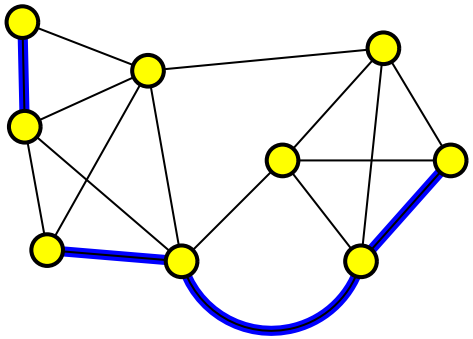

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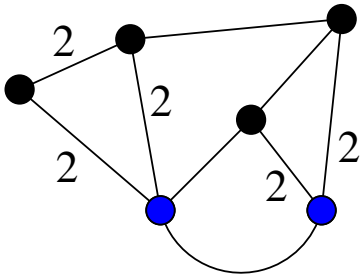

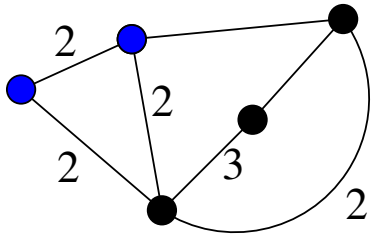

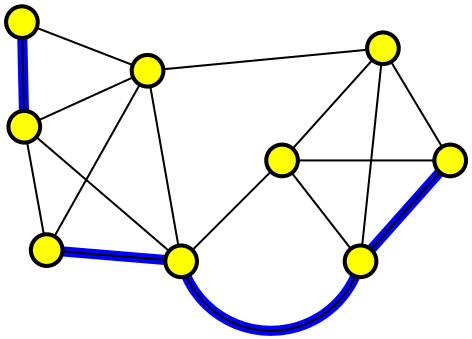

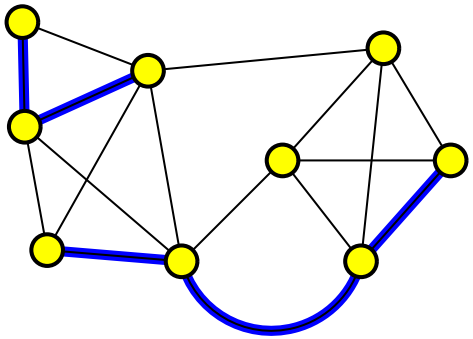

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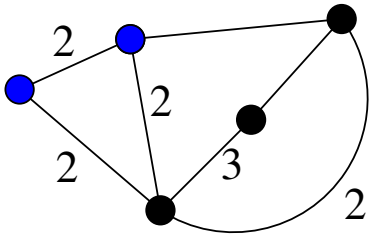

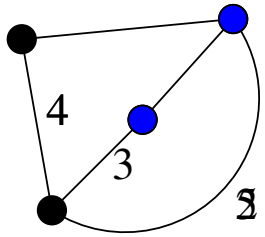

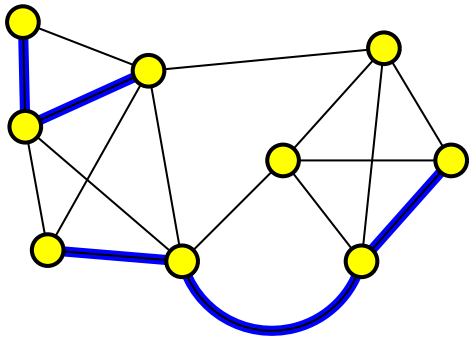

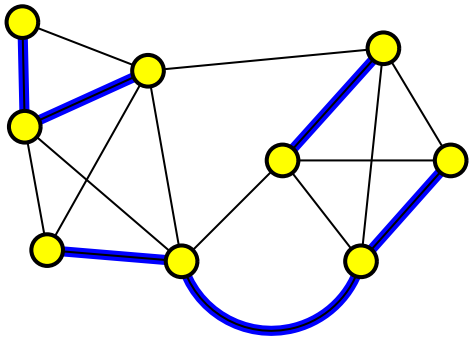

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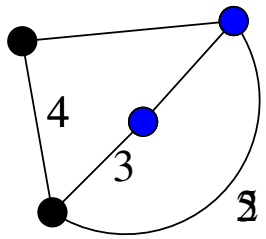

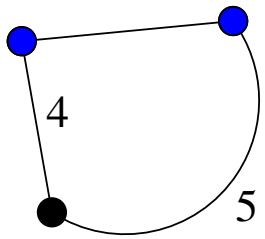

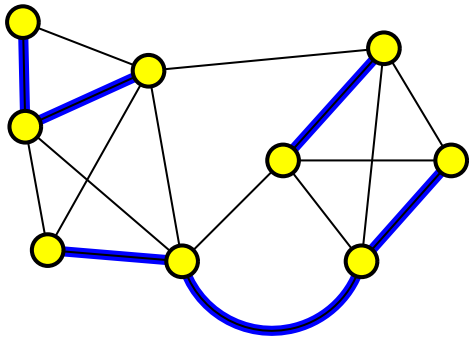

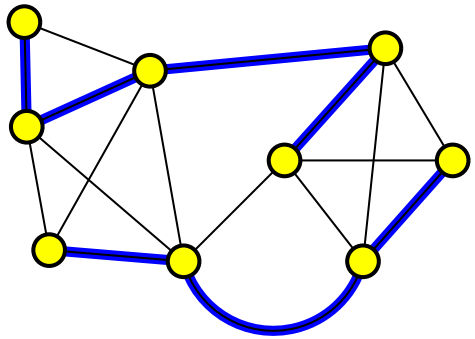

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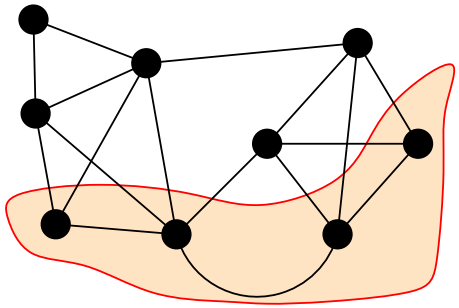

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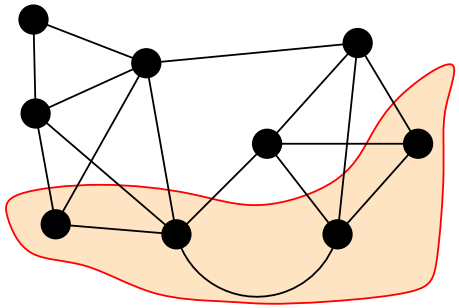

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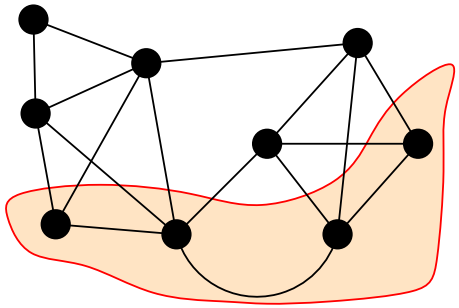

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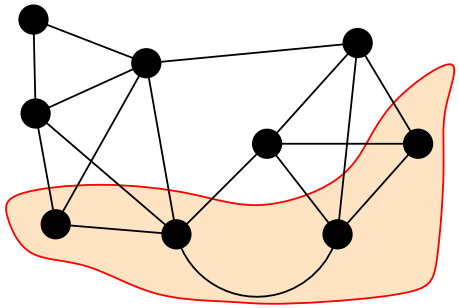

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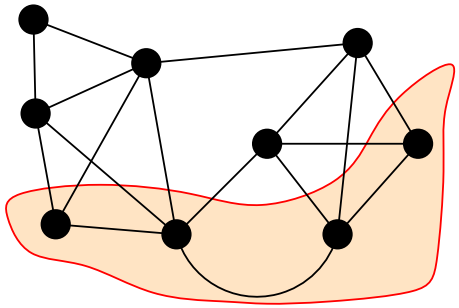

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The algorithm...

Algorithm MinCut($\mathbf{G}$)

$\mathbf{G_0} \leftarrow G$

$i = 0$

while $\mathbf{G}_i$ has more than two vertices **do**

$e_i \leftarrow$ random edge from $\mathbf{E}(\mathbf{G}_i)$

$\mathbf{G}_{i+1} \leftarrow G_i / e_i$

$i \leftarrow i + 1$

Let $(S, V \setminus S)$ be the cut in the original graph corresponding to the single edge in $\mathbf{G}_i$

return $(S, V \setminus S)$.

How to pick a random edge?

Lemma

$X = \{x_1, \ldots, x_n\}$: elements, $\omega(x_i)$: integer positive weight. Pick randomly, in $O(n)$ time, an element $\in X$, with prob picking x_i being $\omega(x_i)/W$, where $W = \sum_{i=1}^{n} \omega(x_i)$.

Proof.

Randomly choose $r \in [0, W]$.
Precompute $\beta_i = \sum_{k=1}^{i} \omega(x_k) = \beta_{i-1} + \omega(x_i)$.
Find first index i, $\beta_{i-1} < r \leq \beta_i$. Return x_i. □

1. Edges have weight...
2. ...compute total weight of each vertex (adjacent edges).
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Lemma

G: *mincut of size* k *and* n *vertices, then* $|\mathbf{E}(\mathbf{G})| \geq \frac{kn}{2}$.

Proof.

Each vertex degree is at least k, otherwise the vertex itself would form a minimum cut of size smaller than k. As such, there are at least $\sum_{v \in V} \mathbf{degree(v)}/2 \geq nk/2$ edges in the graph. □

Lemma...

Lemma

If we pick in random an edge e from a graph $\mathbf{G}$, then with probability at most $\frac{2}{n}$ it belong to the minimum cut.

Proof.

There are at least $nk/2$ edges in the graph and exactly k edges in the minimum cut. Thus, the probability of picking an edge from the minimum cut is smaller then $k/(nk/2) = 2/n$. □

Lemma

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MinCut *outputs the mincut with prob.* $\geq \dfrac{2}{n(n-1)}$.

Proof

1. $\mathcal{E}_i$: event that e_i is not in the minimum cut of $\mathbf{G}_i$.
2. **MinCut** outputs mincut if all the events $\mathcal{E}_0, \ldots, \mathcal{E}_{n-3}$ happen.
3. $\Pr\left[\mathcal{E}_i \,\middle|\, \mathcal{E}_0 \cap \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}\right] \geq 1 - \dfrac{2}{|V(G_i)|} = 1 - \dfrac{2}{n-i}$.

$$\implies \Delta = \Pr[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-3}] = \Pr[\mathcal{E}_0] \cdot \Pr\left[\mathcal{E}_1 \,\middle|\, \mathcal{E}_0\right] \cdot \Pr\left[\mathcal{E}_2 \,\middle|\, \mathcal{E}_0 \cap \mathcal{E}_1\right] \cdot \ldots \cdot \Pr\left[\mathcal{E}_{n-3} \,\middle|\, \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-4}\right]$$

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Proof continued...

As such, we have

$$\begin{aligned}
\Delta &\geq \prod_{i=0}^{n-3}\left(1-\frac{2}{n-i}\right) = \prod_{i=0}^{n-3}\frac{n-i-2}{n-i} \\
&= \frac{n-2}{n} * \frac{n-3}{n-1} * \frac{n-4}{n-2}\cdots\frac{2}{4}\cdot\frac{1}{3} \\
&= \frac{2}{n\cdot(n-1)}.
\end{aligned}$$

Some math restated...

$$\alpha = \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right)\cdots\left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{3}\right)$$

$$= \frac{n-2}{n} \cdot \frac{(n-1)-2}{n-1} \cdot \frac{(n-2)-2}{n-2} \cdots \frac{4-2}{4} \cdot \frac{3-2}{3}$$

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$$= \frac{2}{n(n-1)}$$

Running time analysis...

Observation

MinCut *runs in* $O(n^2)$ *time.*

Observation

The algorithm always outputs a cut, and the cut is not smaller than the minimum cut.

Definition

Amplification: running an experiment again and again till the things we want to happen, with good probability, do happen.

Getting a good probability

MinCutRep: algorithm runs **MinCut** $n(n-1)$ times and return the minimum cut computed.

Lemma

probability **MinCutRep** *fails to return the minimum cut is* < 0.14.

Proof.

MinCut fails to output the mincut in each execution is at most $1 - \frac{2}{n(n-1)}$.

MinCutRep fails, only if all $n(n-1)$ executions of **MinCut** fail.

$\left(1 - \frac{2}{n(n-1)}\right)^{n(n-1)} \leq \exp\left(-\frac{2}{n(n-1)} \cdot n(n-1)\right) =$

$\exp(-2) < 0.14$, since $1 - x \leq e^{-x}$ for $0 \leq x \leq 1$. □

Result

Theorem

One can compute mincut in $O(n^4)$ time with constant probability to get a correct result. In $O(n^4 \log n)$ time the minimum cut is returned with high probability.

Faster algorithm

Why **MinCutRep** needs so many executions?

Probability of failure in first ν iterations is

$$\Pr\left[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{\nu-1}\right] \geq \prod_{i=0}^{\nu-1}\left(1 - \frac{2}{n-i}\right) = \prod_{i=0}^{\nu-1} \frac{n-i-2}{n-i}$$

$$= \frac{n-2}{n} * \frac{n-3}{n-1} * \frac{n-4}{n-2} \cdots$$

$$= \frac{(n-\nu)(n-\nu-1)}{n \cdot (n-1)}.$$

$\Longrightarrow$ $\nu = n/2$: Prob of success $\approx 1/4$.

$\Longrightarrow$ $\nu = n - \sqrt{n}$: Prob of success $\approx 1/n$.

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1. As the graph get smaller probability for bad choice increases.
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Contract...

Contract(G, t) shrinks **G** till it has only t vertices. **FastCut** computes the minimum cut using **Contract**.

Contract(**G**, t)
  while $|(G)| > t$ **do**
    Pick random edge e in **G**.
    G $\leftarrow G/e$
  return **G**

FastCut(**G** $= (V, E)$)
  // **G** -- multi-graph
  $n \leftarrow |V(G)|$
  if $n \leq 6$ **then**
    return mincut **G** (brute force)
  $t \leftarrow \left\lceil 1 + n/\sqrt{2} \right\rceil$
  $H_1 \leftarrow$ **Contract**(G, t)
  $H_2 \leftarrow$ **Contract**(G, t)
  /* **Contract** is randomized */
  $X_1 \leftarrow$ **FastCut**(H_1),
  $X_2 \leftarrow$ **FastCut**(H_2)
  return mincut of X_1 and X_2.

Lemma...

Lemma

The running time of **FastCut**(G) *is* $O(n^2 \log n)$*, where* $n = |V(G)|$.

Proof.

Well, we perform two calls to **Contract**(G, t) which takes $O(n^2)$ time. And then we perform two recursive calls on the resulting graphs. We have:

$$T(n) = O(n^2) + 2T\left(\frac{n}{\sqrt{2}}\right)$$

The solution to this recurrence is $O(n^2 \log n)$ as one can easily (and should) verify. □

Lemma...

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The running time of **FastCut**(G) *is* $O(n^2 \log n)$*, where* $n = |V(G)|$.

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Success at each step

Lemma

Probability that mincut in contracted graph is original mincut is at least $1/2$.

Proof.

Plug in $\nu = n - t = n - \left\lceil 1 + n/\sqrt{2} \right\rceil$ into success probability:

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$$= \frac{\left\lceil 1 + n/\sqrt{2} \right\rceil \left(\left\lceil 1 + n/\sqrt{2} \right\rceil - 1 \right)}{n(n-1)} \geq \frac{1}{2}.$$

□

Probability of success...

Lemma

FastCut *finds the minimum cut with probability larger than* $\Omega(1/\log n)$.

See class notes for a formal proof. We provide a more elegant direct argument shortly.

Amplification

Lemma

Running **FastCut** *repeatedly* $c \cdot \log^2 n$ *times, guarantee that the algorithm outputs mincut with probability* $\geq 1 - 1/n^2$.
c *is a constant large enough.*

Proof.

1. **FastCut** succeeds with prob $\geq c'/\log n$, c' is a constant.
2. ...fails with prob. $\leq 1 - c'/\log n$.
3. ...fails in m reps with prob. $\leq (1 - c'/\log n)^m$. But then $(1 - c'/\log n)^m \leq \left(e^{-c'/\log n}\right)^m \leq e^{-mc'/\log n} \leq \frac{1}{n^2}$, for $m = (2\log^2 n)/c'$.

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Theorem

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One can compute the minimum cut in a graph **G** *with* n *vertices in* $O(n^2 \log^3 n)$ *time. The algorithm succeeds with probability* $\geq 1 - 1/n^2$.

Proof.

We do amplification on **FastCut** by running it $O(\log^2 n)$ times. The running time bound follows from lemma... □

Notes

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