Randomized Algorithms III – Min Cut

Lecture 14
October 13, 2015
14.2: Min cut
**Min cut**

\[ G = (V, E) \]: undirected graph, \( n \) vertices, \( m \) edges.
Interested in cuts in \( G \).

**Definition**

**cut in \( G \):** a partition of \( V \): \( S \) and \( V \setminus S \).

Edges of the cut:

\[ (S, V \setminus S) = \{ uv \mid u \in S, v \in V \setminus S, \text{ and } uv \in E \}, \]

\(|(S, V \setminus S)|\) is size of the cut

**minimum cut / mincut:** cut in graph with min size.
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Some definitions

1. **conditional probability** of $X$ given $Y$ is

$$\Pr\left[ X = x \mid Y = y \right] = \frac{\Pr\left[ (X=x) \cap (Y=y) \right]}{\Pr[ Y = y ]}.$$

$$\Pr\left[ X = x \cap Y = y \right] = \Pr\left[ X = x \mid Y = y \right] \cdot \Pr[ Y = y ].$$

2. $X, Y$ events are **independent**, if

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$$\implies \Pr\left[ X = x \mid Y = y \right] = \Pr\left[ X = x \right].$$
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Lemma

$\mathcal{E}_1, \ldots, \mathcal{E}_n$: $n$ events (not necessarily independent). Then,

$$\Pr\left[ \cap_{i=1}^n \mathcal{E}_i \right] = \Pr[\mathcal{E}_1] \times \Pr[\mathcal{E}_2 | \mathcal{E}_1] \times \Pr[\mathcal{E}_3 | \mathcal{E}_1 \cap \mathcal{E}_2] \times \ldots \times \Pr[\mathcal{E}_n | \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-1}] .$$
Edge contraction...

1. **edge contraction**: $e = xy$ in $G$.
2. ... merge $x, y$ into a single vertex.
3. ... remove self loops.
4. ... parallel edges – multi-graph.
5. ... weights/ multiplicities on the edges.
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Min cut in weighted graph

Edge contraction implemented in $O(n)$ time:

1. Graph represented using adjacency lists.
2. Merging the adjacency lists of the two vertices being contracted.
3. Fix adjacency list of vertices connected to $x, y$. [Use radix sort.]
4. Include edge weight in computing cut weight.
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Cuts under contractions

Observation

1. A cut in $G/xy$ is a valid cut in $G$.
2. There $\exists$ cuts in $G$ are not in $G/xy$.
3. The cut $S = \{x\}$ is not in $G/xy$.
4. $\implies$ size mincut in $G/xy \geq$ mincut in $G$.

Idea: Repeatedly perform edge contractions (benefits: shrink graph)...

Every vertex in contracted graph is a connected component in the original graph.)
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Contraction

(2)
Contraction

(2)

(3)
Contraction

(2)

(3)

(4)
Contraction

(3) $\{x, y\}$

(4) $\{x, y\}$
Contraction

\{x, y\}

(4)
Contraction

\[ \{x, y\} \]

(4)

(5)
Contraction

(5)
Contraction

(7)
Contraction

(7)  

(8)  

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Contraction

(8)

(9)
Contraction

(8)

(9)

(10)
Contraction

(10)

(11)

(12)
Contraction

(11)

(12)

(13)
Contraction

(12)  

(13)
(13)
Contraction

(13)

(14)
Contraction - all together now

(a) (b) (c) (d)

(e) (f) (g) (h)

(i) (j)
Another interpretation
Another interpretation

\begin{itemize}
\item $x$
\item $y$
\end{itemize}
Another interpretation
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But...

1. Not min cut!
2. Contracted wrong edge somewhere...
3. If never contract an edge in the cut...
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5. We might still get min cut even if we contract edge min cut. Why???
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Algorithm MinCut(G)

\( G_0 \leftarrow G \)

\( i = 0 \)

\( \text{while } G_i \text{ has more than two vertices do} \)

\( e_i \leftarrow \text{random edge from } E(G_i) \)

\( G_{i+1} \leftarrow G_i/e_i \)

\( i \leftarrow i + 1 \)

Let \((S, V \setminus S)\) be the cut in the original graph corresponding to the single edge in \( G_i \)

return \((S, V \setminus S)\).
How to pick a random edge?

Lemma

\[ X = \{ x_1, \ldots, x_n \} : \text{elements, } \omega(x_i) : \text{integer positive weight.} \]

Pick randomly, in \( O(n) \) time, an element \( x \in X \), with prob picking \( x_i \) being \( \omega(x_i)/W \), where \( W = \sum_{i=1}^{n} \omega(x_i) \).

Proof.

Randomly choose \( r \in [0, W] \).

Precompute \( \beta_i = \sum_{k=1}^{i} \omega(x_k) = \beta_{i-1} + \omega(x_i) \).

Find first index \( i \), \( \beta_{i-1} < r \leq \beta_i \). Return \( x_i \).

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2. ...compute total weight of each vertex (adjacent edges).
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Lemma

\( G: \text{ mincut of size } k \text{ and } n \text{ vertices, then } |E(G)| \geq \frac{kn}{2}. \)

Proof.

Each vertex degree is at least \( k \), otherwise the vertex itself would form a minimum cut of size smaller than \( k \). As such, there are at least \( \sum_{v \in V} \frac{\text{degree}(v)}{2} \geq \frac{nk}{2} \) edges in the graph.
Lemma

If we pick in random an edge $e$ from a graph $G$, then with probability at most $\frac{2}{n}$ it belong to the minimum cut.

Proof.

There are at least $\frac{nk}{2}$ edges in the graph and exactly $k$ edges in the minimum cut. Thus, the probability of picking an edge from the minimum cut is smaller then $\frac{k}{(nk/2)} = \frac{2}{n}$. \qed
Lemma

**MinCut** outputs the mincut with prob. \[ \geq \frac{2}{n(n - 1)}. \]

Proof

1. \( E_i \): event that \( e_i \) is not in the minimum cut of \( G_i \).
2. **MinCut** outputs mincut if all the events \( E_0, \ldots, E_{n-3} \) happen.

\[ \Pr[E_i \mid E_0 \cap E_1 \cap \ldots \cap E_{i-1}] \geq 1 - \frac{2}{|V(G_i)|} = 1 - \frac{2}{n - i}. \]

\[ \Rightarrow \Delta = \Pr[E_0 \cap \ldots \cap E_{n-3}] = \Pr[E_0] \cdot \Pr[E_1 \mid E_0] \cdot \Pr[E_2 \mid E_0 \cap E_1] \cdot \ldots \cdot \Pr[E_{n-3} \mid E_0 \cap \ldots \cap E_{n-4}] \]
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Proof

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2. **MinCut** outputs mincut if all the events \( \mathcal{E}_0, \ldots, \mathcal{E}_{n-3} \) happen.
3. \[
\Pr\left[ \mathcal{E}_i \bigg| \mathcal{E}_0 \cap \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1} \right] \geq 1 - \frac{2}{|V(G_i)|} = 1 - \frac{2}{n-i}.
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   \[\implies \Delta = \Pr[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-3}] = \Pr[\mathcal{E}_0] \cdot \Pr[\mathcal{E}_1 \bigg| \mathcal{E}_0] \cdot \Pr[\mathcal{E}_2 \bigg| \mathcal{E}_0 \cap \mathcal{E}_1] \cdot \ldots \cdot \Pr[\mathcal{E}_{n-3} \bigg| \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-4}]\]
Proof continued...

As such, we have

\[ \Delta \geq \prod_{i=0}^{n-3} \left( 1 - \frac{2}{n-i} \right) = \prod_{i=0}^{n-3} \frac{n-i-2}{n-i} \]

\[ = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{2}{4} \cdot \frac{1}{3} \]

\[ = \frac{2}{n \cdot (n-1)}. \]
Some math restated...

\[ \alpha = \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{3}\right) \]

\[ = \frac{n - 2}{n} \cdot \frac{(n - 1) - 2}{n - 1} \cdot \frac{(n - 2) - 2}{n - 2} \cdots \frac{4 - 2}{4} \cdot \frac{3 - 2}{3} \]

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\[ = \frac{n - 2}{n} \cdot \frac{n - 3}{n - 1} \cdot \frac{n - 4}{n - 2} \cdots \frac{n - 5}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \]
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\[ \alpha = \left(1 - \frac{2}{n}\right) \left(1 - \frac{2}{n-1}\right) \left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{3}\right) \]

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\[= \frac{n - 2}{n} \cdot \frac{n - 3}{n - 1} \cdot \frac{n - 4}{n - 2} \cdot \frac{n - 5}{n - 3} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \]
Some math restated...

\[ \alpha = \left(1 - \frac{2}{n}\right) \left(1 - \frac{2}{n-1}\right) \left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{3}\right) \]

\[= \frac{n-2}{n} \cdot \frac{(n-1)-2}{n-1} \cdot \frac{(n-2)-2}{n-2} \cdots \frac{4-2}{4} \cdot \frac{3-2}{3} \]

\[= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} \cdots \frac{3}{3} \cdot \frac{2}{4} \cdot \frac{1}{3} \]
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\[= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \cdots \cdot \frac{2}{n-2} \cdot \frac{1}{n-1} \]
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\[ = \frac{2}{n(n-1)} \]
Observation

**MinCut** runs in $O(n^2)$ time.

Observation

The algorithm always outputs a cut, and the cut is not smaller than the minimum cut.

Definition

**Amplification**: running an experiment again and again till the things we want to happen, with good probability, do happen.
Getting a good probability

**MinCutRep**: algorithm runs **MinCut** \(n(n - 1)\) times and return the minimum cut computed.

**Lemma**

The probability **MinCutRep** fails to return the minimum cut is \(< 0.14\).

**Proof.**

**MinCut** fails to output the mincut in each execution is at most \(1 - \frac{2}{n(n-1)}\).

**MinCutRep** fails, only if all \(n(n - 1)\) executions of **MinCut** fail.

\[
\left(1 - \frac{2}{n(n-1)}\right)^{n(n-1)} \leq \exp\left(-\frac{2}{n(n-1)} \cdot n(n-1)\right) = \exp(-2) < 0.14, \text{ since } 1 - x \leq e^{-x} \text{ for } 0 \leq x \leq 1.
\]
Theorem

One can compute mincut in $O(n^4)$ time with constant probability to get a correct result. In $O(n^4 \log n)$ time the minimum cut is returned with high probability.
Faster algorithm

Why **MinCutRep** needs so many executions?

Probability of failure in first $\nu$ iterations is

$$\Pr[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{\nu-1}] \geq \prod_{i=0}^{\nu-1} \left(1 - \frac{2}{n-i}\right) = \prod_{i=0}^{\nu-1} \frac{n-i-2}{n-i}$$

$$= \frac{n-2}{n} \times \frac{n-3}{n-1} \times \frac{n-4}{n-2} \times \ldots$$

$$= \frac{(n-\nu)(n-\nu-1)}{n \cdot (n-1)}.$$ 

$\Rightarrow \quad \nu = n/2$: Prob of success $\approx 1/4$.

$\Rightarrow \quad \nu = n - \sqrt{n}$: Prob of success $\approx 1/n$. 

Why *MinCutRep* needs so many executions?

Probability of failure in first $\nu$ iterations is

$$\Pr\left[ \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{\nu-1} \right] \geq \prod_{i=0}^{\nu-1} \left( 1 - \frac{2}{n-i} \right) = \prod_{i=0}^{\nu-1} \frac{n-i-2}{n-i}$$

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Why \textbf{MinCutRep} needs so many executions?
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Faster algorithm...

Insight

1. As the graph get smaller probability for bad choice increases.
2. Currently do the amplification from the outside of the algorithm.
3. Put amplification directly into the algorithm.
Faster algorithm...

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1. As the graph get smaller probability for bad choice increases.
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Faster algorithm...

**Insight**

1. As the graph gets smaller, the probability of a bad choice increases.
2. Currently, the amplification is done from the outside of the algorithm.
3. Put amplification directly into the algorithm.
**Contract**

**Contract**($G$, $t$) shrinks $G$ till it has only $t$ vertices. **FastCut** computes the minimum cut using **Contract**.

$$\text{FastCut}(G = (V, E))$$

// $G$ -- multi-graph

$n \leftarrow |V(G)|$

if $n \leq 6$ then

\[ t \leftarrow \lceil 1 + n/\sqrt{2} \rceil \]

$H_1 \leftarrow \text{Contract}(G, t)$

$H_2 \leftarrow \text{Contract}(G, t)$

/* **Contract** is randomized */

$X_1 \leftarrow \text{FastCut}(H_1)$,

$X_2 \leftarrow \text{FastCut}(H_2)$

return mincut of $X_1$ and $X_2$. 

**Contract**($G$, $t$)

while $|(G)| > t$ do

Pick random edge $e$ in $G$.

$G \leftarrow G/e$

return $G$
Lemma

The running time of \textbf{FastCut}(G) is \( O(n^2 \log n) \), where \( n = |V(G)| \).

Proof.

Well, we perform two calls to \textbf{Contract}(G, t) which takes \( O(n^2) \) time. And then we perform two recursive calls on the resulting graphs. We have:

\[
T(n) = O(n^2) + 2T\left(\frac{n}{\sqrt{2}}\right)
\]

The solution to this recurrence is \( O(n^2 \log n) \) as one can easily (and should) verify.
The running time of $\text{FastCut}(G)$ is $O(n^2 \log n)$, where $n = |V(G)|$.

Proof.

Well, we perform two calls to $\text{Contract}(G, t)$ which takes $O(n^2)$ time. And then we perform two recursive calls on the resulting graphs. We have:

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The solution to this recurrence is $O(n^2 \log n)$ as one can easily (and should) verify.
Success at each step

**Lemma**

*Probability that mincut in contracted graph is original mincut is at least $1/2$.*

**Proof.**

Plug in $\nu = n - t = n - \left\lceil 1 + n/\sqrt{2} \right\rceil$ into success probability:

$$\Pr \left[ \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-t} \right] \geq$$
Success at each step

Lemma

Probability that mincut in contracted graph is original mincut is at least $\frac{1}{2}$.

Proof.

Plug in $\nu = n - t = n - \left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil$ into success probability:

$$\Pr \left[ \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-t} \right] \geq \frac{t(t-1)}{n \cdot (n-1)}$$
Success at each step

Lemma

*Probability that mincut in contracted graph is original mincut is at least 1/2.*

Proof.

Plug in \( \nu = n - t = n - \left[ 1 + \frac{n}{\sqrt{2}} \right] \) into success probability:

\[
\Pr \left[ E_0 \cap \ldots \cap E_{n-t} \right] \geq \frac{t(t-1)}{n \cdot (n-1)} \geq \frac{\left[ 1 + \frac{n}{\sqrt{2}} \right] \left( \left[ 1 + \frac{n}{\sqrt{2}} \right] - 1 \right)}{n(n-1)} \geq \frac{1}{2}.
\]
Lemma

FastCut finds the minimum cut with probability larger than $\Omega(1/\log n)$.

See class notes for a formal proof. We provide a more elegant direct argument shortly.
Amplification

**Lemma**

Running **FastCut** repeatedly $c \cdot \log^2 n$ times, guarantee that the algorithm outputs mincut with probability $\geq 1 - 1/n^2$. $c$ is a constant large enough.

**Proof.**

1. **FastCut** succeeds with prob $\geq c'/\log n$, $c'$ is a constant.
2. ...fails with prob. $\leq 1 - c'/\log n$.
3. ...fails in $m$ reps with prob. $\leq (1 - c'/\log n)^m$. But then $(1 - c'/\log n)^m \leq (e^{-c'/\log n})^m \leq e^{-mc'/\log n} \leq \frac{1}{n^2}$, for $m = (2 \log^2 n)/c'$. 
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Amplification

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Theorem

One can compute the minimum cut in a graph $G$ with $n$ vertices in $O(n^2 \log^3 n)$ time. The algorithm succeeds with probability $\geq 1 - 1/n^2$.

Proof.

We do amplification on FastCut by running it $O(\log^2 n)$ times. The running time bound follows from lemma...