NEW CS 473: Theory II, Fall 2015

Backward analysis

Lecture 13
October 8, 2015

Some more probability

Lemma
$\mathcal{E}_1, \ldots, \mathcal{E}_n$: n events (not necessarily independent). Then,

$$\mathbf{Pr}\left[\cap_{i=1}^{n} \mathcal{E}_i\right] = \mathbf{Pr}\left[\mathcal{E}_1\right] * \mathbf{Pr}\left[\mathcal{E}_2 \,|\mathcal{E}_1\right] * \mathbf{Pr}\left[\mathcal{E}_3 \,\middle|\, \mathcal{E}_1 \cap \mathcal{E}_2\right] * \ldots * \mathbf{Pr}\left[\mathcal{E}_n \,\middle|\, \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-1}\right].$$

Backward analysis

1. $\mathsf{P} = \langle \mathsf{p}_1, \ldots, \mathsf{p}_n \rangle$ be a random ordering of n distinct numbers.
2. $X_i = 1 \iff \mathsf{p}_i$ is smaller than $\mathsf{p}_1, \ldots, \mathsf{p}_{i-1}$.
3. **Lemma**
   $\mathbf{Pr}[X_i = 1] = 1/i$.

Proof...

Lemma
$\mathbf{Pr}[X_i = 1] = 1/i$.

Proof.

1. Fix elements appearing in $\mathrm{set}(\mathsf{P}_i) = \{s_1, \ldots, s_i\}$.
2. $\mathbf{Pr}\left[\mathsf{p}_i = \min(\mathsf{P}_i) \,\middle|\, \mathrm{set}(\mathsf{P}_i)\right] = 1/i$.

$$\begin{aligned}
&\mathbf{Pr}\left[\mathsf{p}_i = \min(\mathsf{P}_i)\right] \\
&= \sum_{S \subseteq \mathsf{P}, |S| = i} \mathbf{Pr}\left[\mathsf{p}_i = \min(\mathsf{P}_i) \,\middle|\, \mathrm{set}(\mathsf{P}_i) = S\right] \mathbf{Pr}[S] \\
&= \sum_{S \subseteq \mathsf{P}, |S| = i} \frac{1}{i} \mathbf{Pr}[S] = \frac{1}{i}.
\end{aligned}$$

$\square$

of times...

...the minimum changes in a random permutation...

Theorem

In a random permutation of n distinct numbers, the minimum of the prefix changes in expectation $\ln n + 1$ *times.*

Proof.

1. $Y = \sum_{i=1}^{n} X_i$.
2. $\mathbf{E}[Y] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} 1/i$ $\leq \ln n + 1$.

$\square$

High probability

Lemma

$\Pi = \pi_1 \ldots \pi_n$: *random permutation of* $\{1, \ldots, n\}$. X_i: *indicator variable if* π_i *is the smallest number in* $\{\pi_1, \ldots, \pi_i\}$, *for* $\forall i$.
Then $Z = \sum_{i=1}^{n} X_i = O(\log n)$., *w.h.p. (i.e.,* $\geq 1 - 1/n^{O(1)}$*).*

proof

1. $\mathcal{E}_i$: the event that $X_i = 1$, for $i = 1, \ldots, n$.
2. Claim: $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are independent.
3. Generate permutation: Randomly pick a permutation of the given numbers, set first number to be π_n.
4. Next, pick a random permutation of the remaining numbers and set the first number as π_{n-1} in output permutation.
5. Repeat this process till we generate the whole permutation.

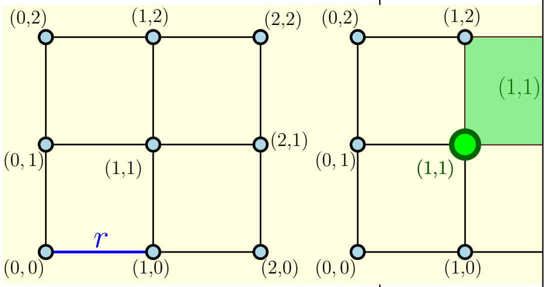
Proof continued...

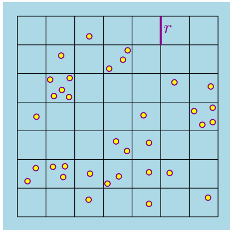
1. For any indices $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, and observe that $\mathbf{Pr}\left[\mathcal{E}_{i_k} \,\middle|\, \mathcal{E}_{i_1} \cap \ldots \cap \mathcal{E}_{i_{k-1}}\right] = \mathbf{Pr}\left[\mathcal{E}_{i_k}\right]$,
2. ..because $\mathcal{E}_{i_1}$ determined after all $\mathcal{E}_{i_2}, \ldots, \mathcal{E}_k$.
3. By induction: $\mathbf{Pr}\left[\mathcal{E}_{i_1} \cap \mathcal{E}_{i_2} \cap \ldots \cap \mathcal{E}_{i_k}\right] =$ $\mathbf{Pr}\left[\mathcal{E}_{i_1} \,\middle|\, \mathcal{E}_{i_2} \cap \ldots \cap \mathcal{E}_{i_k}\right] \mathbf{Pr}[\mathcal{E}_{i_2} \cap \ldots \cap \mathcal{E}_{i_k}] =$ $\mathbf{Pr}\left[\mathcal{E}_{i_1}\right] \mathbf{Pr}\left[\mathcal{E}_{i_2} \cap \mathcal{E}_{i_3} \cap \ldots \cap \mathcal{E}_{i_k}\right] = \prod_{j=1}^{k} \mathbf{Pr}\left[\mathcal{E}_{i_j}\right] =$ $\prod_{j=1}^{k} \frac{1}{i_j}$.
4. $\implies$ variables $X_1, \ldots, X_n$ are independent.
5. Result readily follows from Chernoff's inequality. $\blacksquare$

Finding the closest pair of points

Grids...

1. r: Side length of grid cell.
2. Grid cell IDed by pair of integers.
3. Constant time to determine a point p's grid cell id:
   $\boxed{id(p) = (\lfloor p_x/r \rfloor, \lfloor p_y/r \rfloor)}$
4. Limited use of the floor function (but no word packing tricks).
5. Use hashing on (grid) points.
6. Store points in grid...
   ...in linear time.

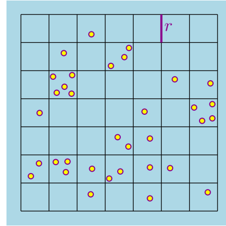




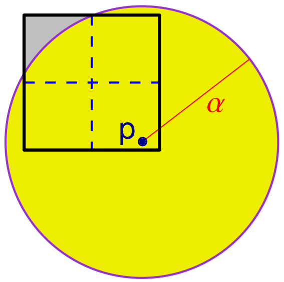
Storing point set in grid/hash-table...

Hashing:

1. Non-empty grid cells
2. For non-empty grid cell:
   List of points in it.
3. For a grid cell:
   Its neighboring cells.



Closet pair in a square



Lemma

Let $\mathbf{P}$ be a set of points contained inside a square $\square$, such that the sidelength of $\square$ is $\alpha = \mathcal{CP}(\mathbf{P})$. Then $|\mathbf{P}| \leq 4$.

Proof.

Partition $\square$ into four equal squares $\square_1, \ldots, \square_4$.
Each square diameter $\sqrt{2}\alpha/2 < \alpha$.
... contain at most one point of $\mathbf{P}$; that is, the disk of radius α centered at a point $\mathbf{p} \in \mathbf{P}$ completely covers the subsquare containing it; see the figure on the right.
$\mathbf{P}$ can have four points if it is the four corners of $\square$. $\square$

Verify closet pair

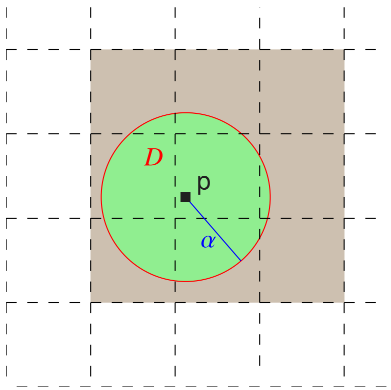
Lemma

$\mathbf{P}$*: set of n points in the plane. α: distance. Verify in linear time whether $\mathcal{CP}(\mathbf{P}) < \alpha$, $\mathcal{CP}(\mathbf{P}) = \alpha$, or $\mathcal{CP}(\mathbf{P}) > \alpha$.*

proof

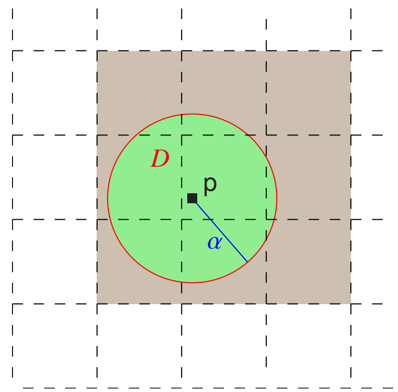
Indeed, store the points of $\mathbf{P}$ in the grid $\mathbf{G}_\alpha$. For every non-empty grid cell, we maintain a linked list of the points inside it. Thus, adding a new point $\mathbf{p}$ takes constant time. Specifically, compute $\mathrm{id}(\mathbf{p})$, check if $\mathrm{id}(\mathbf{p})$ already appears in the hash table, if not, create a new linked list for the cell with this ID number, and store $\mathbf{p}$ in it. If a linked list already exists for $\mathrm{id}(\mathbf{p})$, just add $\mathbf{p}$ to it. This takes $O(n)$ time overall. Now, if any grid cell in $\mathbf{G}_\alpha(\mathbf{P})$ contains more than, say, 4 points of $\mathbf{P}$, then it must be that the $\mathcal{CP}(\mathbf{P}) < \alpha$, by previous lemma.

Proof continued



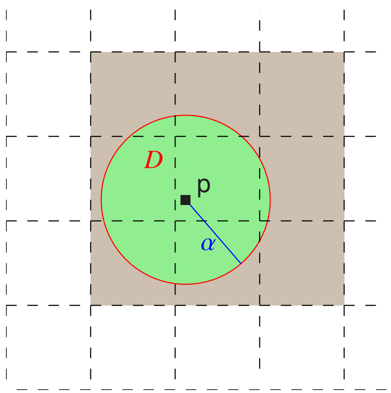
1. When insert a point $\mathbf{p}$: fetch all the points of $\mathbf{P}$ in cluster of $\mathbf{P}$
2. Takes constant time.
3. If there is a point closer to $\mathbf{p}$ than α that was already inserted, then it must be stored in one of these $\mathbf{9}$ cells.
4. Now, each one of those cells must contain at most $\mathbf{4}$ points of $\mathbf{P}$ by prev lemma.
5. Otherwise, already stopped since $\mathcal{CP}(\cdot) < \alpha$.

Proof continued



1. S set of all points in cluster.
2. $|S| \leq 9 \cdot 4 = O(1)$.
3. Compute closest point to $\mathbf{p}$ in S. $O(1)$ time.
4. If $\mathbf{d}(\mathbf{p}, S) < \alpha$, we stop; otherwise, continue to next point.
5. Correctness: '$\mathcal{CP}(\mathbf{P}) < \alpha$' returned only if such pair found.

Proof continued



1. Assume $\mathbf{p}$ and $\mathbf{q}$: realizing closest pair.
2. $\|\mathbf{p} - \mathbf{q}\| = \mathcal{CP}(\mathbf{P}) < \alpha$.
3. When later point (say $\mathbf{p}$) inserted, the set S would contain $\mathbf{q}$.
4. algorithm would stop and return '$\mathcal{CP}(\mathbf{P}) < \alpha$'.
5. ■

New algorithm

1. Pick a random permutation of the points of $\mathbf{P}$.
2. $\langle \mathbf{p}_1, \ldots, \mathbf{p}_n \rangle$ be this permutation.
3. $\alpha_2 = \|\mathbf{p}_1 - \mathbf{p}_2\|$.
4. Insert points into the closet-pair distance verifying data-structure.
5. α_i: the closest pair distance in the set $\mathbf{P}_i = \{\mathbf{p}_1, \ldots, \mathbf{p}_i\}$, for $i = 2, \ldots, n$.
6. ith iteration:
   - 6.1 if $\alpha_i = \alpha_{i-1}$. insertion takes constant time.
   - 6.2 If $\alpha_i < \alpha_{i-1}$ then: know new closest pair distance α_i.
   - 6.3 rebuild the grid, and reinsert the i points of $\mathbf{P}_i$ from scratch into the grid $\mathbf{G}_{\alpha_i}$. Takes $O(i)$ time.
7. Returns the number α_n and points realizing it.

Weak analysis...

Lemma

Let t be the number of different values in the sequence $\alpha_2, \alpha_3, \ldots, \alpha_n$. Then $\mathbf{E}[t] = O(\log n)$. As such, in expectation, the above algorithm rebuilds the grid $O(\log n)$ times.

proof

1. $X_i = 1 \iff \alpha_i < \alpha_{i-1}$.
2. $\mathbf{E}[X_i] = \mathbf{Pr}[X_i = 1]$ and $t = \sum_{i=3}^n X_i$.
3. $\mathbf{Pr}[X_i = 1] = \mathbf{Pr}[\alpha_i < \alpha_{i-1}]$.
4. Backward analysis. Fix P_i.
5. $\mathsf{q} \in \mathsf{P}_i$ is **critical** if $\mathcal{CP}(\mathsf{P}_i \setminus \{\mathsf{q}\}) > \mathcal{CP}(\mathsf{P}_i)$.
6. No critical points, then $\alpha_{i-1} = \alpha_i$ and then $\mathbf{Pr}[X_i = 1] = 0$.

Proof continued...

1. If one critical point, then $\mathbf{Pr}[X_i = 1] = 1/i$.
2. Assume two critical points and let p, q be this unique pair of points of P_i realizing $\mathcal{CP}(\mathsf{P}_i)$.
3. $\alpha_i < \alpha_{i-1} \iff \mathsf{p}$ or q is p_i.
4. $\mathbf{Pr}[X_i = 1] = 2/i$.
5. Cannot be more than two critical points.
6. Linearity of expectations: $\mathbf{E}[t] = \mathbf{E}\left[\sum_{i=3}^n X_i\right] = \sum_{i=3}^n \mathbf{E}[X_i] \leq \sum_{i=3}^n 2/i = O(\log n)$.
7. $\blacksquare$

Expected linear time analysis...

Theorem

P: set of n points in the plane. Compute the closest pair of P in expected linear time.

Proof.

1. $X_i = 1 \iff \alpha_i \neq \alpha_{i-1}$.
2. Running time is proportional to $R = 1 + \sum_{i=3}^n (1 + X_i \cdot i)$.
3. $\mathbf{E}[R] = \mathbf{E}\left[1 + \sum_{i=3}^n (1 + X_i \cdot i)\right] \leq n + \sum_{i=3}^n \mathbf{E}[X_i] \cdot i \leq n + \sum_{i=3}^n i \cdot \mathbf{Pr}[X_i = 1] \leq n + \sum_{i=3}^n i \cdot \frac{2}{i} \leq 3n$, by linearity of expectation and since $\mathbf{E}[X_i] = \mathbf{Pr}[X_i = 1] \leq 2/i$.
4. Expected running time of the algorithm is $O(\mathbf{E}[R]) = O(n)$. $\blacksquare$

$\square$

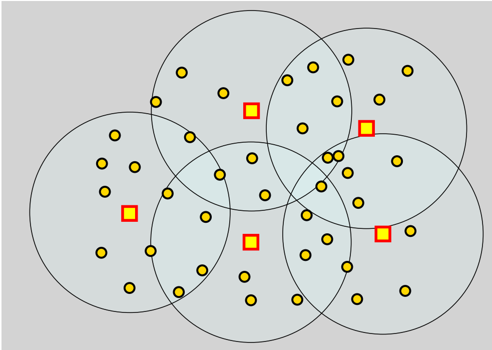
Nets

The Main Tool

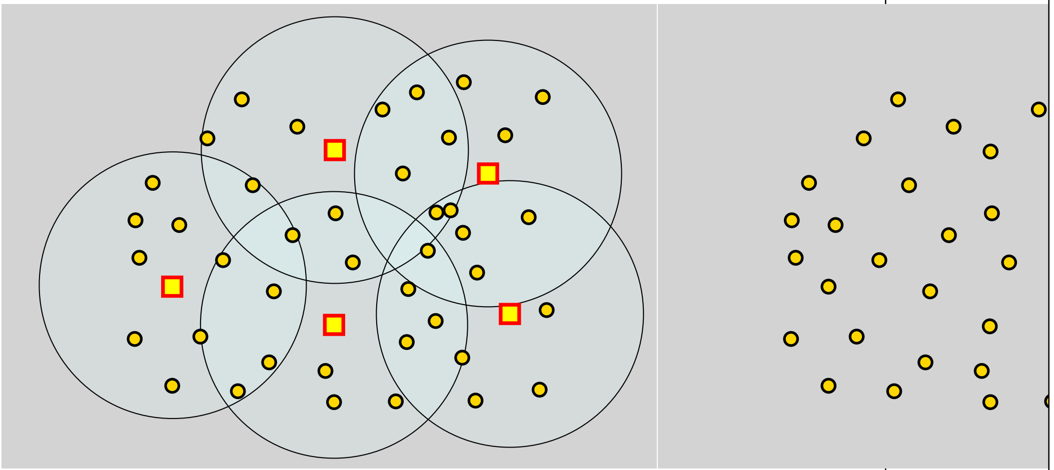
r-net

$N \subseteq \mathsf{P}$ is an r-net if

- Every point in P has distance $< r$ to a point in N
- For any two $\mathsf{p}, \mathsf{q} \in N$, we have $d(\mathsf{p}, \mathsf{q}) \geq r$.

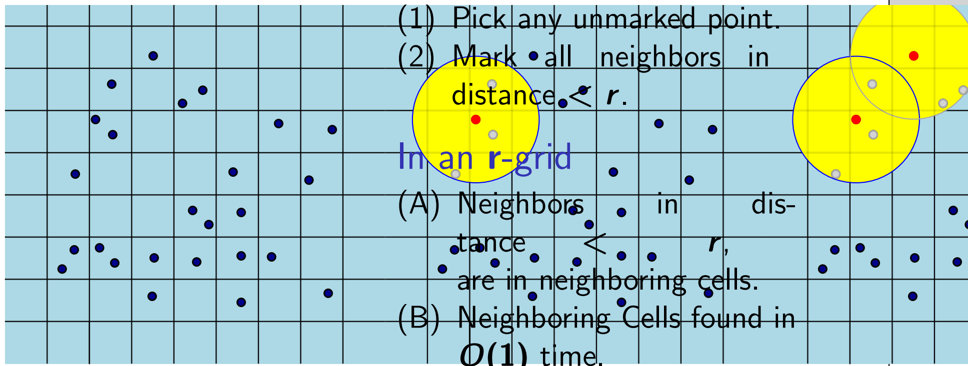


Computing an r-net



Application of Grids: Computing nets

...in linear time



Repeatedly:

(1) Pick any unmarked point.

(2) Mark all neighbors in distance $< r$.

In an **r**-grid

(A) Neighbors in distance $< r$, are in neighboring cells.

(B) Neighboring Cells found in $O(1)$ time.

(C) Cells contain lists of points.

Input

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: edge-weighted.
2. n vertices and m edges.
3. Task: compute an ordering $\pi = \langle \pi_1, \ldots, \pi_n \rangle$ of vertices.
4. $\forall v \in \mathbf{V} \qquad L_v$:
   $\pi_i \in Ł_v, \iff \pi_i$ closet vertex to v in prefix $\langle \pi_1, \ldots, \pi_i \rangle$.
5. Example: Streaming scenario - install servers in a network.
   .... every client in t network needs to know its closest server.
6. ... client needs to maintain its current closest server.
7. How min total size of lists? $\mathcal{L} = \sum_{v \in \mathbf{V}} |L_v|$.

Algorithm

1. $\pi_1, \ldots, \pi_n$: random permutation of $\mathbf{V}$ of $\mathbf{G}$.
2. $\forall v \in \mathbf{V} \qquad \delta(v) = +\infty$.
3. For $i = 1$ to n do:
   3.1 $\delta(\pi_i)$ to $\mathbf{0}$,
   3.2 start Dijkstra from the ith vertex π_i.
   3.3 Dijkstra propagates to u only if improves current distance.
   3.4 Update $\delta(u)$ to $\mathbf{d_G}(\pi_i, u) \iff \mathbf{d_G}(\pi_i, u) < \delta'(u)$
   $\delta'(u)$: value before this iteration started.
   3.5 If $\delta(u)$ updated: add π_i to L_u.

Performance

Lemma

Algorithm computes a permutation π, such that:

1. $\mathbf{E}\Big[|\mathcal{L}|\Big] = O(n \log n)$.
2. *Expected running time* $O\Big((n \log n + m) \log n\Big)$.
3. $n = |\mathbf{V}(\mathbf{G})|$ *and* $m = |\mathbf{E}(\mathbf{G})|$.
4. *Both bounds also hold with high probability.*

Proof

1. Fix a vertex $v \in \mathbf{V} = \{v_1, \ldots, v_n\}$.
2. $U = \{\mathrm{d_G}(v, v_1), \ldots, \mathrm{d_G}(v, v_n)\}$.
3. $\mathrm{d_G}(v, \pi_1), \ldots, \mathrm{d_G}(v, \pi_n)$: random permutation of U.
4. By lemma seen π min changes $O(\log n)$ times in expectations + high prob.
5. $|L_v| = O(\log n)$ in expectation + high probability.
6. Running time:
   6.1 For $uv \in \mathbf{E}(\mathbf{G})$: $\delta(u)$ or $\delta(v)$ changes $O(\log n)$ times.
   6.2 uv gets visited $O(\log n)$ times by all "Disjkstras",
   6.3 Overall running time $O(n \log^2 n + m \log n)$:
      6.3.1 $O(n \log n)$ changes in $\delta(\cdot)$.
      6.3.2 n: **delete-min** operations
      6.3.3 Edge triggers $O(\log n)$ **decrease-key** operations.
      6.3.4 time(**decrease-key**)=$O(1)$
      time(**delete-min**)=$O(\log n)$.
      (Fibonacci heaps).

Computing r-net in sparse graphs.

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a weighted graph with n vertices and m edges, let $r > 0$.
2. π_i: ith vertex in a random permutation of $\mathbf{V}$.
3. $\forall v \in \mathbf{V} : \delta(v) := +\infty$.
4. Test whether $\delta(\pi_i) \geq r$, if so:
   4.1 Add π_i to the resulting net $\mathcal{N}$.
   4.2 Set $\delta(\pi_i)$ to zero.
   4.3 Perform Dijkstra's algorithm starting from π_i,
   Occupy a vertex u only if improve distance: $\delta(u)$.
   4.4 If a vertex u is expanded: $\delta(u)$: computed distance from π_i, and relax the edges adjacent to u in the graph.

Correctness

Lemma

The set $\mathcal{N}$ is an r-net in $\mathbf{G}$.

Proof.

1. End: $\forall v \in \mathbf{V}$: $\delta(v) < r$.
2. By induction: if $\ell = \delta(v)$, for some vertex v, then the distance of v to the set $\mathcal{N}$ is at most ℓ.
3. Every two points in $\mathcal{N}$ have distance $\geq r$. Indeed, when the algorithm handles vertex $v \in \mathcal{N}$, its distance from all the vertices currently in $\mathcal{N}$ is $\geq r$.

$\square$

Correctness continued...

Lemma

Consider an execution of the algorithm, and any vertex $v \in \mathbf{V}$. The expected number of times the algorithm updates the value of $\delta(v)$ during its execution is $O(\log n)$, and more strongly the number of updates is $O(\log n)$ with high probability.

Proof...

1. Assume all distances in $\mathbf{G}$ are distinct.
2. S_i: set of all vertices $x \in \mathbf{V}$, such that:
   (A) $d(x, v) < d(v, \Pi_i)$, where $\Pi_i = \{\pi_1, \ldots, \pi_i\}$.
   (B) If $\pi_{i+1} = x$ then $\delta(v)$ would change in the $(i+1)$th iteration.
3. $s_i = |S_i|$. Observe $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n$, and $|S_n| = 0$.
4. $\mathcal{E}_{i+1}$: event that $\delta(v)$ changed in iteration $(i+1)$ (**active** iteration).
5. $(i+1)$ iteration active: $\pi_{i+1} \in S_i$.
6. π_{i+1}: uniform distribution over the vertices of S_i.

Proof continued...

1. $\mathcal{E}_{i+1}$ happens then $s_{i+1} \leq s_i/2$, with probability $\geq 1/2$.
2. iteration is **lucky**.
3. After $O(\log n)$ lucky iterations set S_i empty: Done.
4. $\mathcal{E}_1, \ldots, \mathcal{E}_n$: Independent.
5. By Chernoff inequality, after $c \log n$ active iterations, at least $\lceil \log_2 n \rceil$ iterations lucky. with high probability.

■

Correctness continued...

Lemma

Given a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, with n vertices and m edges, the above algorithm computes an r-net of $\mathbf{G}$ in $O((n+m)\log n)$ expected time.

Proof.

By above lemma, the two δ values associated with the endpoints of an edge get updated $O(\log n)$ times, in expectation, during the algorithm's execution. As such, a single edge creates $O(\log n)$ decrease-key operations in the heap maintained by the algorithm. Each such operation takes constant time if we use Fibonacci heaps to implement the algorithm. □