Approximation Algorithms III

Lecture 10

September 24, 2015

10.1: Subset Sum

Subset Sum

Instance: $X = \{x_1, \ldots, x_n\}$ – n integer positive numbers, t - target number
Question: $\exists$ subset of X s.t. sum of its elements is t?

Assume $x_1, \ldots, x_n$ are all $\leq n$. Then this problem can be solved in

(A) The problem is still **NP-Hard**, so probably exponential time.
(B) $O(n^3)$.
(C) $2^{O(\log^2 n)}$.
(D) $O(n \log n)$.
(E) None of the above.

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M: Max value input numbers.

SolveSubsetSum (X, t, M)
    $b[0 \ldots Mn] \Leftarrow$ **false**
        // $b[x]$ is **true** if x can be
            // realized by subset of X.
    $b[0] \leftarrow$ **true**.
    for $i = 1, \ldots, n$ **do**
        for $j = Mn$ down to x_i **do**
            $b[j] \leftarrow B[j - x_i] \vee B[j]$
    return $B[t]$

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R.T. $O(Mn^2)$.

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Efficient algorithm???

1. Algorithm solving **Subset Sum** in $O(Mn^2)$.
2. M might be prohibitly large...
3. if $M = 2^n \implies$ algorithm is not polynomial time.
4. **Subset Sum** is **NPC**.
5. Still want to solve quickly even if M huge.
6. Optimization version:

Subset Sum Optimization

Instance: (X, t): A set X of n positive integers, and a target number t.
Question: The largest number γ_{opt} one can represent as a subset sum of X which is smaller or equal to t.

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Subset Sum

2-approximation

Lemma

1. (X, t); *Given instance of* **Subset Sum**. $\gamma_{\mathrm{opt}} \leq t$: *Opt.*
2. $\implies$ *Compute legal subset with sum* $\geq \gamma_{\mathrm{opt}}/2$.
3. *Running time* $O(n \log n)$.

Proof.

1. Sort numbers in X in decreasing order.
2. Greedily - add numbers from largest to smallest (if possible).
3. s: Generates sum.
4. u: First rejected number. s': sum before rejection.
5. $s' > u > 0$, $s' < t$, and $s' + u > t \implies$
   $t < s' + u < s' + s' = 2s' \implies s' \geq t/2$.

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10.1.1: On the complexity of ε-approximation algorithms

Polynomial Time Approximation Schemes

Definition (PTAS)

PROB: **Max**imization problem.

$\varepsilon > 0$: approximation parameter.

$\mathcal{A}(I, \varepsilon)$ is a **polynomial time approximation scheme** (**PTAS**) for **PROB**:

1. $\forall I$: $(1 - \varepsilon) \left|\mathbf{opt}(I)\right| \leq \left|\mathcal{A}(I, \varepsilon)\right| \leq \left|\mathbf{opt}(I)\right|$,
2. $|\mathbf{opt}(I)|$: opt price,
3. $|\mathcal{A}(I, \varepsilon)|$: price of solution of $\mathcal{A}$.
4. $\mathcal{A}$ running time polynomial in n for fixed ε.

For minimization problem:

$|\mathbf{opt}(I)| \leq |\mathcal{A}(I, \varepsilon)| \leq (1 + \varepsilon)|\mathbf{opt}(I)|$.

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Polynomial Time Approximation Schemes

1. Example: Approximation algorithm with running time $O(n^{1/\varepsilon})$ is a PTAS.
   Algorithm with running time $O(1/\varepsilon^n)$ is not.
2. Fully polynomial...

Definition (FPTAS)

An approximation algorithm is **fully polynomial time approximation scheme** (**FPTAS**) if it is a PTAS, and its running time is polynomial both in n and $1/\varepsilon$.

3. Example: PTAS with running time $O(n^{1/\varepsilon})$ is not a FPTAS.
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Approximating Subset Sum

Subset Sum Approx

Instance: (X, t, ε): A set X of n positive integers, a target number t, and parameter $\varepsilon > 0$.
Question: A number z that one can represent as a subset sum of X, such that $(1 - \varepsilon)\gamma_{\text{opt}} \le z \le \gamma_{\text{opt}} \le t$.

Approximating Subset Sum

Looking again at the exact algorithm

ExactSubsetSum(**S**, t)
  $n \leftarrow |S|$
  $P_0 \leftarrow \{0\}$
  for $i = 1 \dots n$ **do**
    $P_i \leftarrow P_{i-1} \cup (P_{i-1} + x_i)$
    Remove from P_i all elements $> t$

  return largest element in P_n

1. $S = \{a_1, \dots, a_n\}$
   $x + S = \{a_1 + x, a_2 + x, \dots a_n + x\}$
2. Lists might explode in size.

Trim the lists...

L': Inc. sorted list of numbers

```
Trim(L', δ)
   L = ⟨y_1 ... y_m⟩
   curr ← y_1
   L_out ← {y_1}
   for i = 2...m do
      if y_i > curr · (1 + δ)
         Append y_i to L_out
         curr ← y_i
   return L_out
```

Definition

For two positive real numbers $z \leq y$, the number y is a δ-approximation to z if

$$\frac{y}{1+\delta} \leq z \leq y.$$

Observation

If $x \in L'$ then there exists a number $y \in L_{out}$ such that $y \leq x \leq y(1+\delta)$, where $L_{out} \leftarrow \textbf{Trim}(L', \delta)$.

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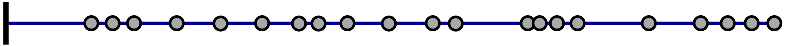
```
ApproxSubsetSum(S, t)
    // S = {x_1, ..., x_n},
    //        x_1 ≤ x_2 ≤ ... ≤ x_n
    n ← |S|, L_0 ← {0},
    δ = ε/2n
    for i = 1 ... n do
        E_i ← L_{i-1} ∪ (L_{i-1} + x_i)
        L_i ← Trim(E_i, δ)
        Remove from L_i elems > t.

    return largest element in L_n
```

E_i: Computed by merging two sorted lists in linear time.

Understanding trimming



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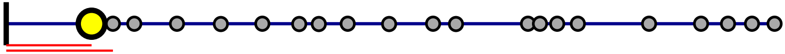

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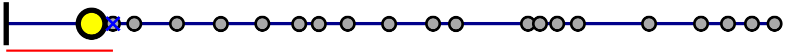

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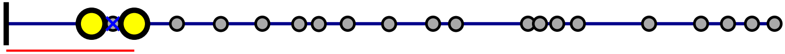

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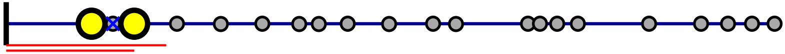

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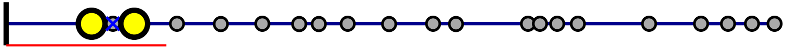

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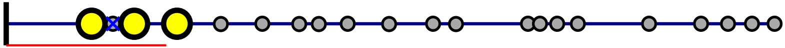

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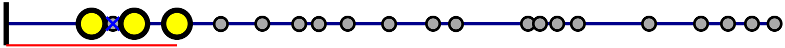

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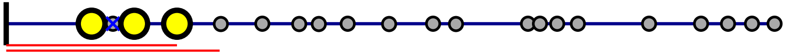

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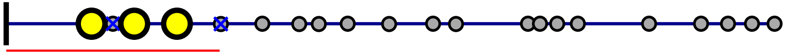

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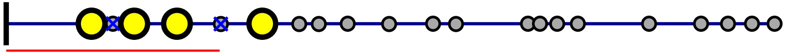

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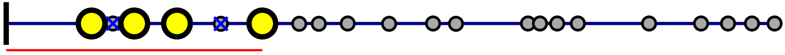

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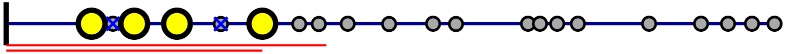

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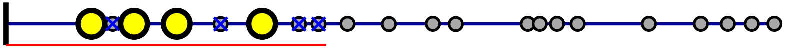

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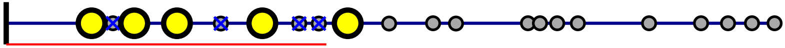

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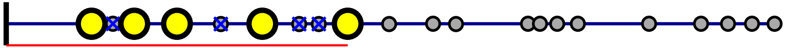

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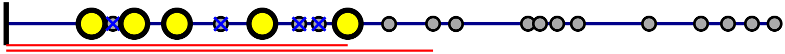

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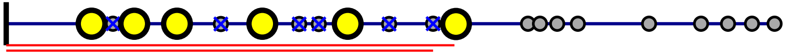

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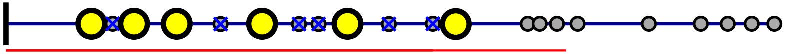

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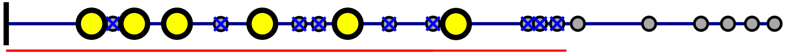

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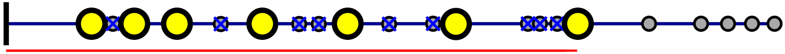

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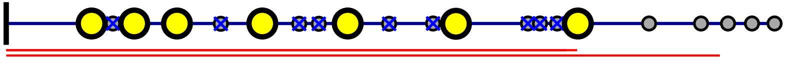

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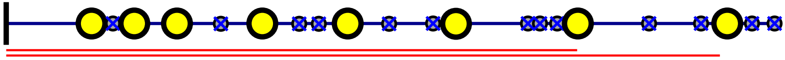

Remark

1. Can assume that trimmed lists L_i are sorted...
2. Algorithm: $E_i \leftarrow L_{i-1} \cup (L_{i-1} + x_i)$
3. So, this is just copy, shift, and merge of two sorted lists.
4. ... resulting in a sorted lest.
5. takes linear time in size of lists.

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Analysis

1. E_i list generated by algorithm in ith iteration.
2. P_i: list of numbers (no trimming).

Claim

For any $x \in P_i$ there exists $y \in L_i$ such that $y \leq x \leq (1+\delta)^i y$.

Proof

1. If $x \in P_1$ then follows by observation above.
2. If $x \in P_{i-1} \implies$ (induction) $\exists y' \in L_{i-1}$ s.t. $y' \leq x \leq (1+\delta)^{i-1} y'$.
3. By observation $\exists y \in L_i$ s.t. $y \leq y' \leq (1+\delta)y$, As such,

$$y \leq y' \leq x \leq (1+\delta)^{i-1} y' \leq (1+\delta)^i y.$$

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Proof continued

1. If $x \in P_i \setminus P_{i-1} \implies x = \alpha + x_i$, for some $\alpha \in P_{i-1}$.
2. By induction, $\exists \alpha' \in L_{i-1}$ s.t. $\alpha' \leq \alpha \leq (1+\delta)^{i-1}\alpha'$.
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10.1.1.1: Running time

Running time of ApproxSubsetSum

Lemma

For $x \in [0, 1]$*, it holds* $\exp(x/2) \leq (1 + x)$.

Lemma

For $0 < \delta < 1$*, and* $x \geq 1$*, we have*

$$\log_{1+\delta} x \leq \frac{2 \ln x}{\delta} = O\left(\frac{\ln x}{\delta}\right).$$

See notes for a proof of lemmas.

Running time of ApproxSubsetSum

Observation

In a list generated by **Trim**, *for any number* x, *there are no two numbers in the trimmed list between* x *and* $(1+\delta)x$.

Lemma

$|L_i| = O\Big((n/\varepsilon)\log n\Big)$, *for* $i = 1, \ldots, n$.

Running time of ApproxSubsetSum

Proof.

1. $L_{i-1} + x_i \subseteq [x_i, ix_i]$.
2. Trimming $L_{i-1} + x_i$ results in list of size

$$\log_{1+\delta} \frac{ix_i}{x_i} = O\left(\frac{\ln i}{\delta}\right) = O\left(\frac{\ln n}{\delta}\right),$$

3. Now, $\delta = \varepsilon / 2n$, and

$$|L_i| \leq |L_{i-1}| + O\left(\frac{\ln n}{\delta}\right) \leq |L_{i-1}| + O\left(\frac{n \ln n}{\varepsilon}\right)$$
$$= O\left(\frac{n^2 \log n}{\varepsilon}\right).$$

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Running time of ApproxSubsetSum

Lemma

The running time of **ApproxSubsetSum** *is* $O\left(\frac{n^3}{\varepsilon}\log n\right)$.

Proof.

1. Running time of **ApproxSubsetSum** dominated by total length of $L_1, \ldots, L_n$.
2. Above lemma implies
$$\sum_i |L_i| = O\left(n \times \frac{n^2}{\varepsilon}\log n\right) = O\left(\frac{n^3}{\varepsilon}\log n\right).$$
3. **Trim** runs in time proportional to size of lists.
4. Overall, R.T. $O\left(\frac{n^3}{\varepsilon}\log n\right)$.

□

ApproxSubsetSum

Theorem

ApproxSubsetSum *returns* $u \leq t$*, s.t.* $\frac{\gamma_{\mathrm{opt}}}{1+\varepsilon} \leq u \leq \gamma_{\mathrm{opt}} \leq t$,
γ_{opt}*: opt solution.*
Running time is $O((n^3/\varepsilon)\log n)$.

Proof.

1. Running time from above.
2. $\gamma_{\mathrm{opt}} \in P_n$: optimal solution.
3. $\exists z \in L_n$, such that $z \leq \mathbf{opt} \leq (1+\delta)^n z$
4. $(1+\delta)^n = (1+\varepsilon/2n)^n \leq \exp\left(\frac{\varepsilon}{2}\right) \leq 1+\varepsilon$, since $1+x \leq e^x$ for $x \geq 0$.
5. $\gamma_{\mathrm{opt}}/(1+\varepsilon) \leq z \leq \mathbf{opt} \leq t$.

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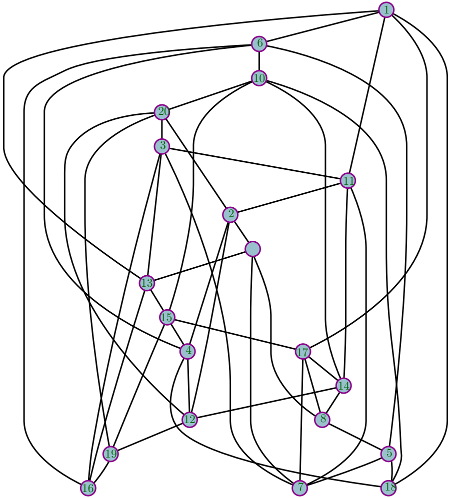
10.2: Maximal matching

Maximal matching

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$
2. Compute maximal matching...
3. $X \subseteq \mathbf{E}$ which is maxim**al** and independent.
4. Maximal $=$ can not improved by adding an edge.
5. Maximum $=$ largest possible set among all possible sets.
6. Computing the maximum is hard then computing maximal solution.
7. Q: Find maximal matching quickly and of large size...

An example of the greedy algorithm...



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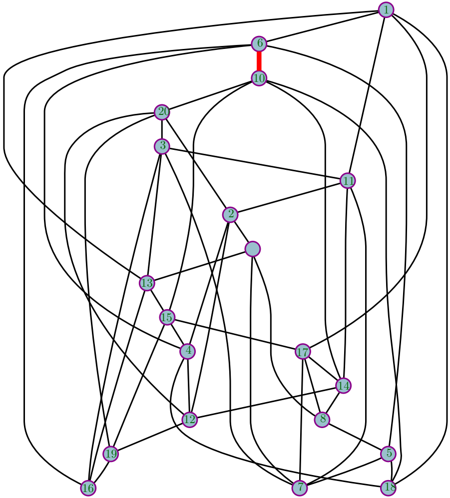

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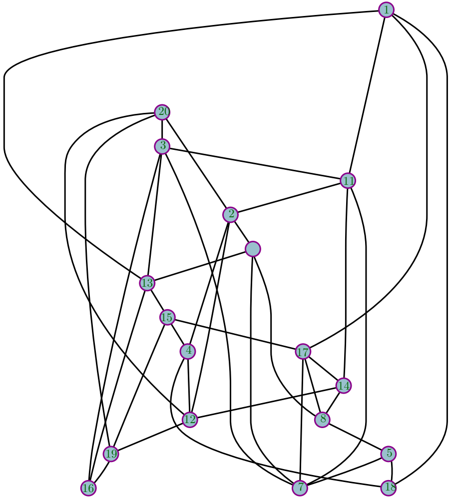

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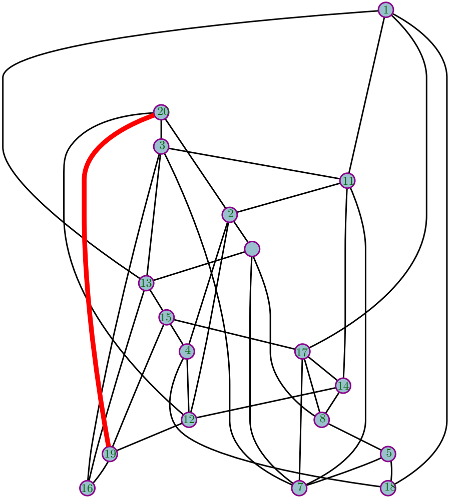

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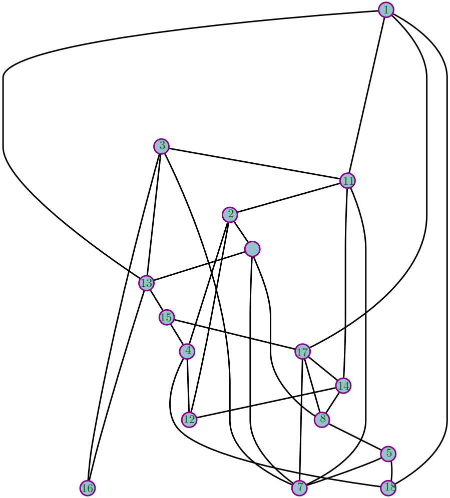

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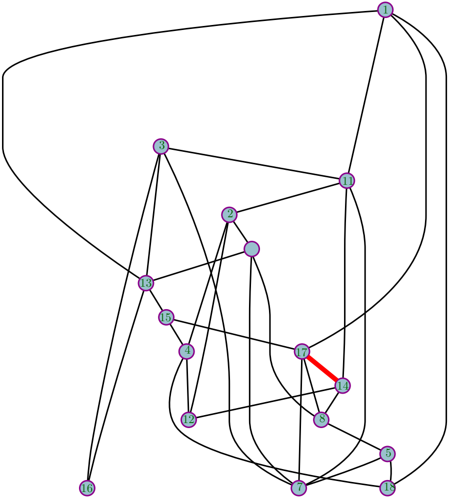

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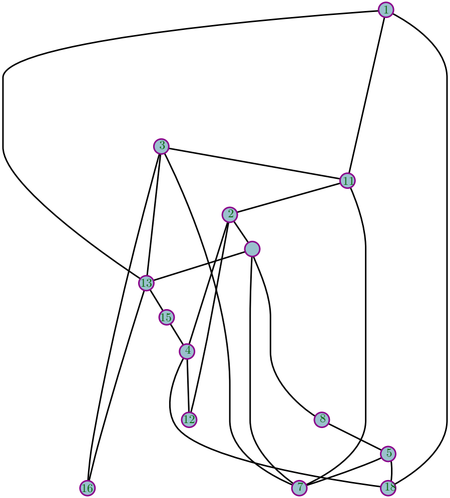

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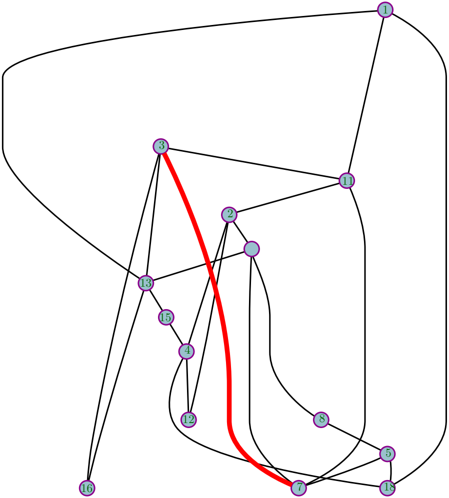

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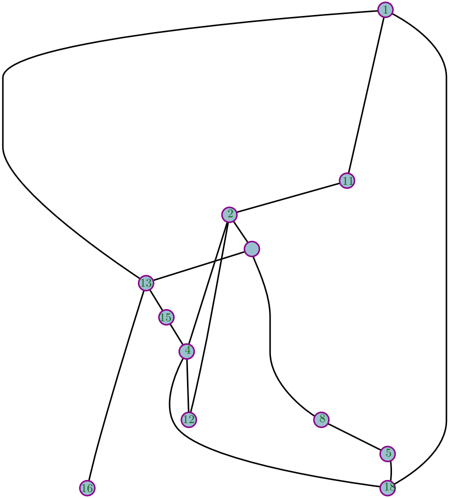

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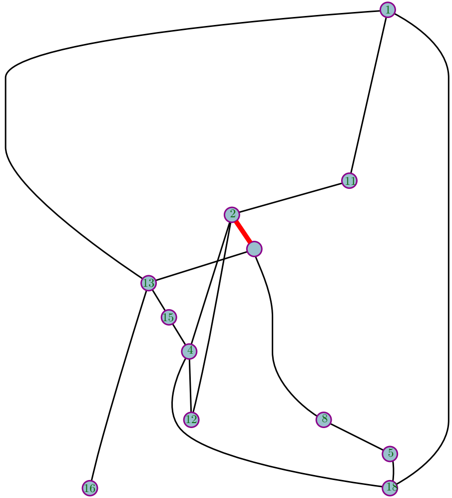

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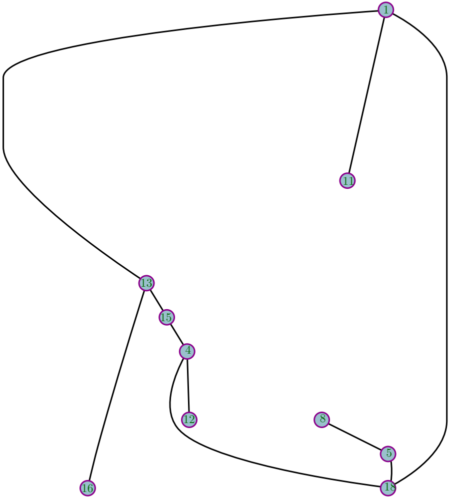

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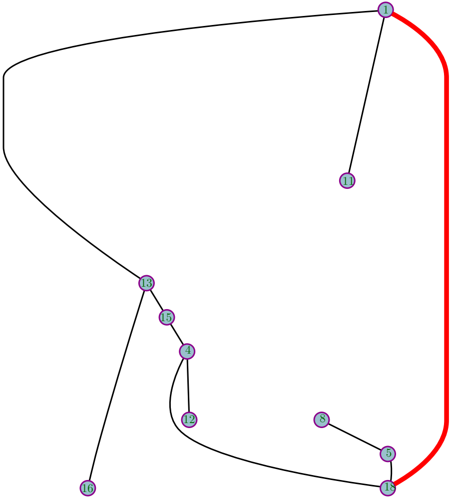

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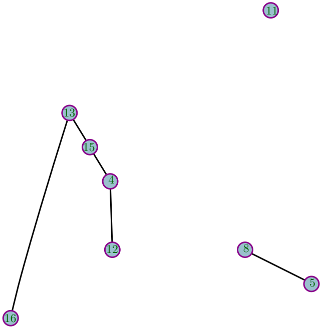

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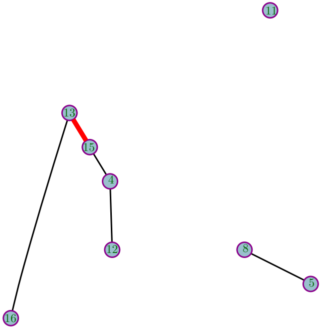

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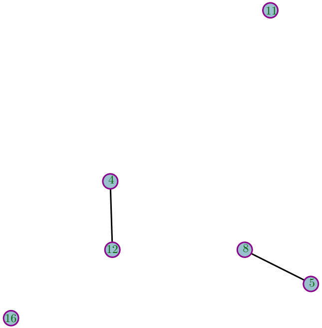

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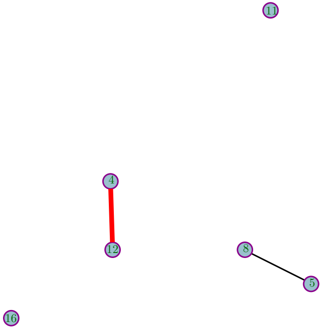

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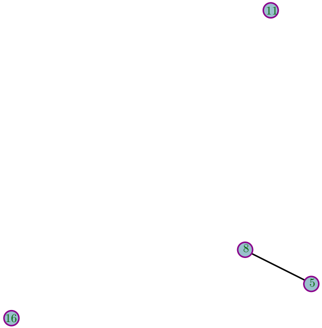

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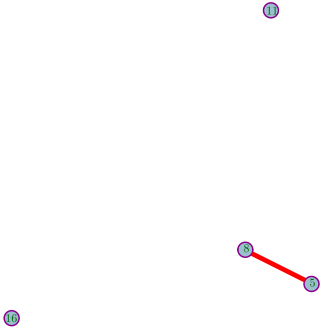

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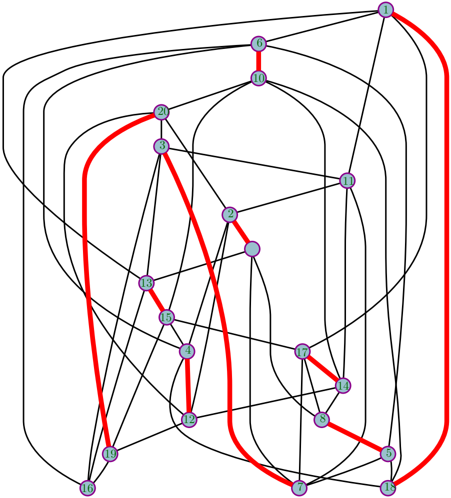

Maximal matching: Algorithm

1. Algorithm: Repeatedly pick an arbitrary edge and remove it.
2. M: Generated matching. X: Maximal matching.
3. Clearly a maximal matching...
4. This is a 2-approximation to the maximum matching.
5. Because...
6. Every edge in M "kills" two edges of X in the worst case.

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Maximal matching: Result

Theorem

Given a graph **G** *one can compute in* $O(n+m)$ *time, a maximal matching with at least* $|X|/2$ *edges, where* X *is the size of the maximum (optimal) matching.*

10.2.1: Bin packing

Bin Packing

Instance: v: Bin size. $S = \{\alpha_1, \ldots, \alpha_n\}$: n items
α_i: size of ith item.
Target: Find min # B, and a decomposition $S_1, \ldots, S_B$ of S, such that $\forall j \quad \sum_{x \in S_j} \leq v$.

1. $\cup_i S_i = S$ and $\forall i \neq j \quad S_i \cap S_j = \emptyset$.
2. **NP-Hard** from **Partition**.
3. **NP-Hard** to approximate within $3/2$.
4. Natural problem...
5. How to approximate?
6. First fit: Have a row of bins, insert items greedily into the first bin that fits them.
7. First fit decreasing: Sort the elements first...

Bin packing

Problem definition

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Bin Packing

Instance: v: Bin size. $S = \{\alpha_1, \ldots, \alpha_n\}$: n items
α_i: size of ith item.
Target: Find min # B, and a decomposition $S_1, \ldots, S_B$ of S, such that $\forall j \quad \sum_{x \in S_j} \leq v$.

1. $\cup_i S_i = S$ and $\forall i \neq j \quad S_i \cap S_j = \emptyset$.
2. **NP-Hard** from **Partition**.
3. **NP-Hard** to approximate within $3/2$.
4. Natural problem...
5. How to approximate?
6. First fit: Have a row of bins, insert items greedily into the first bin that fits them.
7. First fit decreasing: Sort the elements first...

Bin packing: First fit

Analysis

Lemma

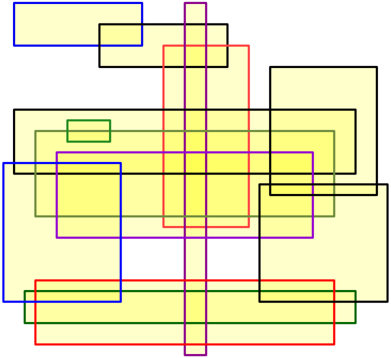
First fit is a 2-approximation.

Proof.

Observe that only one bin can have less than $v/2$ content in it... □

10.3: Independent set of axis-parallel rectangles

An example

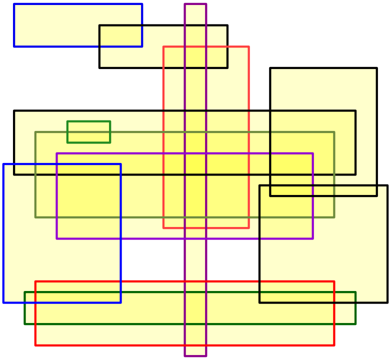


Input

Assume: Open rectangles.

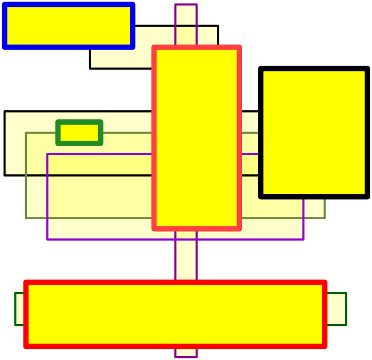
Independent set of rectangles.

An example



Input

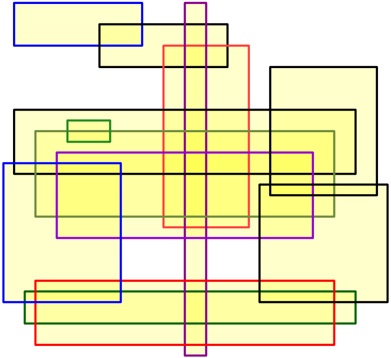
Assume: Open rectangles.



Independent set of rectangles.

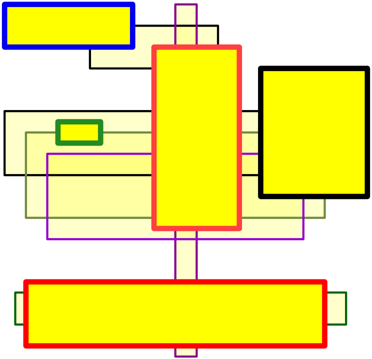
Independent set of rectangles

Algorithm: Divide & Conquer



Independent set of rectangles

Algorithm: Divide & Conquer



Independent set of rectangles

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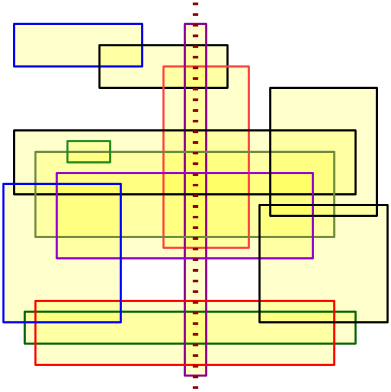

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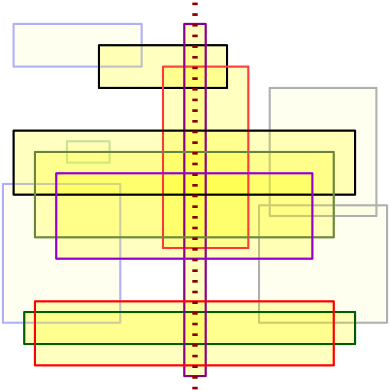

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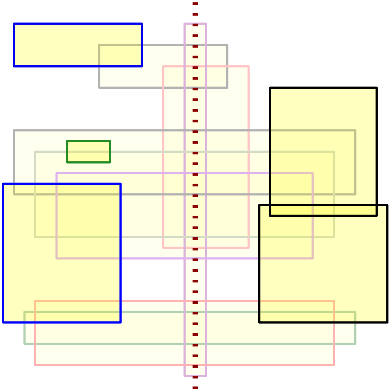

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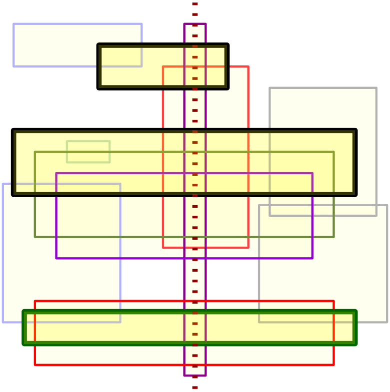

Independent set of rectangles

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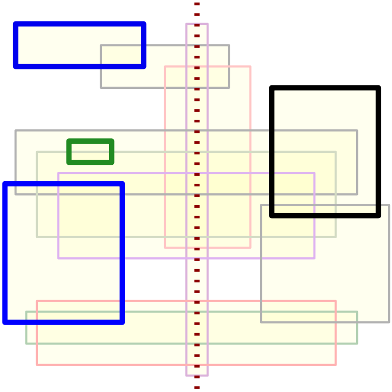

Independent set of rectangles

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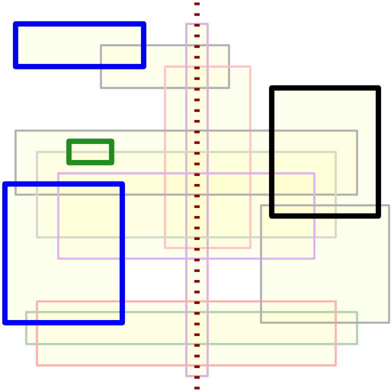

Independent set of rectangles

Algorithm: Divide & Conquer

$\mathcal{R}$: A set of axis parallel rectangles.

```
RectIndep(R):
  if |R| ≤ 10 then
    Solve by brute force
    return size of solution
  x_M:  Median of right x-coordinate of rects in R
  ℓ:  Vertical line through x_M.
  R_M:  Rects of R intersecting ℓ
  R_L, R_R:  Rectangles in R left/ right of ℓ.
  S_L ⇐ RectIndep(R_L)
  S_R ⇐ RectIndep(R_R)
  S_M ⇐ compute opt solution for R_M (intervals!)
  return max(S_M, S_L + S_R)
```

Analysis

1. If $S_M \geq \mathrm{Opt}/(2\lg n)$... done.
2. $\mathrm{Opt}_L + \mathrm{Opt}_R \geq (1 - 1/(2\lg n))\mathrm{Opt}$.
3. By induction: $S_L \geq \mathrm{Opt}_L/(2\lg(n/2))$ and $S_R \geq \mathrm{Opt}_R/(2\lg(n/2))$.
4. $S_L + S_R \geq \frac{(1-1/(2\lg n))\mathrm{Opt}}{2\lg(n/2)}$
5. $$\frac{(1-1/(2\lg n))}{2\lg(n/2)} = \frac{1}{2\lg n - 2} - \frac{1}{(2\lg n)(2\lg n - 2)}$$ $$\geq \frac{2\lg n - 1}{(2\lg n)(2\lg n - 2)} \geq \frac{2\lg n - 2}{(2\lg n)(2\lg n - 2)} \geq \frac{1}{2\lg n}.$$
6. Conclude: If $S_M \leq \mathrm{Opt}/(2\lg n)$, then $S_L + S_R \geq \mathrm{Opt}/(2\lg n)$.
7. Algorithm is $2\lg n$ approximation.

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