NEW CS 473: Theory II, Fall 2015

Approximation Algorithms III

Lecture 10
September 24, 2015

Subset Sum

Subset Sum

Instance: $X = \{x_1, \ldots, x_n\}$ – n integer positive numbers, t - target number
Question: $\exists$ subset of X s.t. sum of its elements is t?

Assume $x_1, \ldots, x_n$ are all $\leq n$. Then this problem can be solved in

- **(A)** The problem is still **NP-Hard**, so probably exponential time.
- **(B)** $O(n^3)$.
- **(C)** $2^{O(\log^2 n)}$.
- **(D)** $O(n \log n)$.
- **(E)** None of the above.

Subset Sum

Subset Sum

Instance: $X = \{x_1, \ldots, x_n\}$ – n integer positive numbers, t - target number
Question: $\exists$ subset of X s.t. sum of its elements is t?

M: Max value input numbers.

R.T. $O(Mn^2)$.

```
SolveSubsetSum (X, t, M)
  b[0...Mn] ⇐ false
    // b[x] is true if x can be
          // realized by subset of X.
  b[0] ← true.
  for i = 1, ..., n do
    for j = Mn down to x_i do
      b[j] ← B[j − x_i] ∨ B[j]
  return B[t]
```


Subset Sum

Efficient algorithm???

1. Algorithm solving **Subset Sum** in $O(Mn^2)$.
2. M might be prohibitly large...
3. if $M = 2^n \implies$ algorithm is not polynomial time.
4. **Subset Sum** is **NPC**.
5. Still want to solve quickly even if M huge.
6. Optimization version:

Subset Sum Optimization

Instance: (X, t): A set X of n positive integers, and a target number t.
Question: The largest number γ_{opt} one can represent as a subset sum of X which is smaller or equal to t.

Subset Sum

2-approximation

Lemma

1. (X, t); *Given instance of* **Subset Sum**. $\gamma_{\text{opt}} \leq t$: *Opt.*
2. $\implies$ *Compute legal subset with sum* $\geq \gamma_{\text{opt}}/2$.
3. *Running time* $O(n \log n)$.

Proof.

1. Sort numbers in X in decreasing order.
2. Greedily - add numbers from largest to smallest (if possible).
3. s: Generates sum.
4. u: First rejected number. s': sum before rejection.
5. $s' > u > 0$, $s' < t$, and $s' + u > t \implies$
   $t < s' + u < s' + s' = 2s' \implies s' \geq t/2$.

Polynomial Time Approximation Schemes

Definition ( )

PROB: **Max**imization problem.
$\varepsilon > 0$: approximation parameter.
$\mathcal{A}(I, \varepsilon)$ is a **polynomial time approximation scheme** (**PTAS**) for **PROB**:

1. $\forall I$: $(1-\varepsilon)\left|\text{opt}(I)\right| \leq \left|\mathcal{A}(I,\varepsilon)\right| \leq \left|\text{opt}(I)\right|$,
2. $|\text{opt}(I)|$: opt price,
3. $|\mathcal{A}(I, \varepsilon)|$: price of solution of $\mathcal{A}$.
4. $\mathcal{A}$ running time polynomial in n for fixed ε.

For minimization problem:
$|\text{opt}(I)| \leq |\mathcal{A}(I, \varepsilon)| \leq (1+\varepsilon)|\text{opt}(I)|$.

Polynomial Time Approximation Schemes

1. Example: Approximation algorithm with running time $O(n^{1/\varepsilon})$ is a PTAS.
   Algorithm with running time $O(1/\varepsilon^n)$ is not.
2. Fully polynomial...

Definition ( )

An approximation algorithm is **fully polynomial time approximation scheme** (**FPTAS**) if it is a PTAS, and its running time is polynomial both in n and $1/\varepsilon$.

3. Example: PTAS with running time $O(n^{1/\varepsilon})$ is not a FPTAS.
4. Example: PTAS with running time $O(n^2/\varepsilon^3)$ is a FPTAS.

Approximating Subset Sum

Subset Sum Approx

Instance: (X, t, ε): A set X of n positive integers, a target number t, and parameter $\varepsilon > 0$.
Question: A number z that one can represent as a subset sum of X, such that $(1-\varepsilon)\gamma_{\text{opt}} \leq z \leq \gamma_{\text{opt}} \leq t$.

Approximating Subset Sum

Looking again at the exact algorithm

```
ExactSubsetSum(S, t)
  n ← |S|
  P_0 ← {0}
  for i = 1 ... n do
      P_i ← P_{i-1} ∪ (P_{i-1} + x_i)
      Remove from P_i all elements > t

  return largest element in P_n
```

1. $S = \{a_1, \ldots, a_n\}$
   $x + S = \{a_1 + x, a_2 + x, \ldots a_n + x\}$
2. Lists might explode in size.

Trim the lists...

L': Inc. sorted list of numbers

```
Trim(L', δ)
  L = ⟨y_1 ... y_m⟩
  curr ← y_1
  L_out ← {y_1}
  for i = 2 ... m do
    if y_i > curr · (1 + δ)
      Append y_i to L_out
      curr ← y_i
  return L_out
```

Definition

For two positive real numbers $z \leq y$, the number y is a δ-approximation to z if $\frac{y}{1+\delta} \leq z \leq y$.

Observation

If $x \in L'$ then there exists a number $y \in L_{out}$ such that $y \leq x \leq y(1+\delta)$, where $L_{out} \leftarrow$ Trim(L', δ).

Trim the lists...

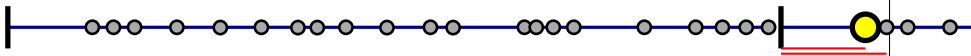
```
Trim(L', δ)
  L = ⟨y_1 ... y_m⟩
  curr ← y_1
  L_out ← {y_1}
  for i = 2 ... m do
    if y_i > curr · (1 + δ)
      Append y_i to L_out
      curr ← y_i
  return L_out
```

```
ApproxSubsetSum(S, t)
// S = {x_1, ..., x_n},
//        x_1 ≤ x_2 ≤ ... ≤ x_n
n ← |S|, L_0 ← {0},
δ = ε/2n
for i = 1 ... n do
   E_i ← L_{i-1} ∪ (L_{i-1} + x_i)
   L_i ← Trim(E_i, δ)
   Remove from L_i elems > t.

return largest element in L_n
```

E_i: Computed by merging two sorted lists in linear time.

Understanding trimming



Remark

1. Can assume that trimmed lists L_i are sorted...
2. Algorithm: $E_i \leftarrow L_{i-1} \cup (L_{i-1} + x_i)$
3. So, this is just copy, shift, and merge of two sorted lists.
4. ... resulting in a sorted lest.
5. takes linear time in size of lists.

Analysis

1. E_i list generated by algorithm in ith iteration.
2. P_i: list of numbers (no trimming).

Claim

For any $x \in P_i$ there exists $y \in L_i$ such that $y \leq x \leq (1+\delta)^i y$.

Proof

1. If $x \in P_1$ then follows by observation above.
2. If $x \in P_{i-1} \implies$ (induction) $\exists y' \in L_{i-1}$ s.t. $y' \leq x \leq (1+\delta)^{i-1} y'$.
3. By observation $\exists y \in L_i$ s.t. $y \leq y' \leq (1+\delta)y$, As such,

$$y \leq y' \leq x \leq (1+\delta)^{i-1} y' \leq (1+\delta)^i y.$$

Proof continued

Proof continued

1. If $x \in P_i \setminus P_{i-1} \implies x = \alpha + x_i$, for some $\alpha \in P_{i-1}$.
2. By induction, $\exists \alpha' \in L_{i-1}$ s.t. $\alpha' \leq \alpha \leq (1+\delta)^{i-1}\alpha'$.
3. Thus, $\alpha' + x_i \in E_i$.
4. $\exists x' \in L_i$ s.t. $x' \leq \alpha' + x_i \leq (1+\delta)x'$.
5. Thus,
   $x' \leq \alpha' + x_i \leq \alpha + x_i = x \leq (1+\delta)^{i-1}\alpha' + x_i \leq (1+\delta)^{i-1}(\alpha' + x_i) \leq (1+\delta)^i x'$. ∎

Running time of ApproxSubsetSum

Lemma

For $x \in [0,1]$, it holds $\exp(x/2) \leq (1+x)$.

Lemma

For $0 < \delta < 1$, and $x \geq 1$, we have

$$\log_{1+\delta} x \leq \frac{2 \ln x}{\delta} = O\left(\frac{\ln x}{\delta}\right).$$

See notes for a proof of lemmas.

Running time of ApproxSubsetSum

Observation

In a list generated by **Trim**, *for any number* x, *there are no two numbers in the trimmed list between* x *and* $(1+\delta)x$.

Lemma

$|L_i| = O\Big((n/\varepsilon)\log n\Big)$, *for* $i = 1, \ldots, n$.

Running time of ApproxSubsetSum

Proof.

1. $L_{i-1} + x_i \subseteq [x_i, ix_i]$.
2. Trimming $L_{i-1} + x_i$ results in list of size

$$\log_{1+\delta} \frac{ix_i}{x_i} = O\left(\frac{\ln i}{\delta}\right) = O\left(\frac{\ln n}{\delta}\right),$$

3. Now, $\delta = \varepsilon/2n$, and

$$|L_i| \leq |L_{i-1}| + O\left(\frac{\ln n}{\delta}\right) \leq |L_{i-1}| + O\left(\frac{n \ln n}{\varepsilon}\right)$$
$$= O\left(\frac{n^2 \log n}{\varepsilon}\right).$$

□

Running time of ApproxSubsetSum

Lemma

The running time of **ApproxSubsetSum** *is* $O\left(\frac{n^3}{\varepsilon}\log n\right)$.

Proof.

1. Running time of **ApproxSubsetSum** dominated by total length of $L_1, \ldots, L_n$.
2. Above lemma implies

$$\sum_i |L_i| = O\left(n \times \frac{n^2}{\varepsilon}\log n\right) = O\left(\frac{n^3}{\varepsilon}\log n\right).$$

3. **Trim** runs in time proportional to size of lists.
4. Overall, R.T. $O\left(\frac{n^3}{\varepsilon}\log n\right)$.

□

ApproxSubsetSum

Theorem

ApproxSubsetSum *returns* $u \leq t$, *s.t.*
$\frac{\gamma_{\text{opt}}}{1+\varepsilon} \leq u \leq \gamma_{\text{opt}} \leq t$,
γ_{opt}: *opt solution.*
Running time is $O\big((n^3/\varepsilon)\log n\big)$.

Proof.

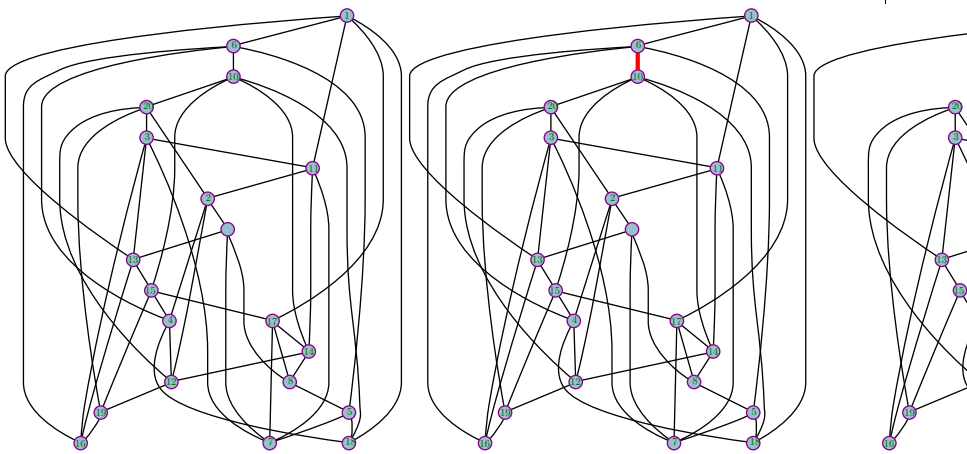
1. Running time from above.
2. $\gamma_{\text{opt}} \in P_n$: optimal solution.
3. $\exists z \in L_n$, such that $z \leq \textbf{opt} \leq (1+\delta)^n z$
4. $(1+\delta)^n = (1+\varepsilon/2n)^n \leq \exp\left(\frac{\varepsilon}{2}\right) \leq 1+\varepsilon$, since $1 + x \leq e^x$ for $x \geq 0$.
5. $\gamma_{\text{opt}}/(1+\varepsilon) \leq z \leq \textbf{opt} \leq t$.

□

Maximal matching

1. $\mathbf{G = (V, E)}$
2. Compute maximal matching...
3. $\boldsymbol{X} \subseteq \mathbf{E}$ which is maxim**al** and independent.
4. Maximal = can not improved by adding an edge.
5. Maximum = largest possible set among all possible sets.
6. Computing the maximum is hard then computing maximal solution.
7. Q: Find maximal matching quickly and of large size...

An example of the greedy algorithm...



Maximal matching: Algorithm

1. Algorithm: Repeatedly pick an arbitrary edge and remove it.
2. $\boldsymbol{M}$: Generated matching. $\boldsymbol{X}$: Maximal matching.
3. Clearly a maximal matching...
4. This is a $\mathbf{2}$-approximation to the maximum matching.
5. Because...
6. Every edge in $\boldsymbol{M}$ "kills" two edges of $\boldsymbol{X}$ in the worst case.

Maximal matching: Result

Theorem

Given a graph $\mathbf{G}$ *one can compute in* $\boldsymbol{O(n+m)}$ *time, a maximal matching with at least* $|\boldsymbol{X}|/\mathbf{2}$ *edges, where* $\boldsymbol{X}$ *is the size of the maximum (optimal) matching.*

Bin packing

Problem definition

Bin Packing

Instance: v: Bin size. $S = \{\alpha_1, \ldots, \alpha_n\}$: n items α_i: size of ith item.
Target: Find min # B, and a decomposition $S_1, \ldots, S_B$ of S, such that $\forall j \quad \sum_{x \in S_j} \leq v$.

1. $\cup_i S_i = S$ and $\forall i \neq j \quad S_i \cap S_j = \emptyset$.
2. **NP-Hard** from **Partition**.
3. **NP-Hard** to approximate within $3/2$.
4. Natural problem...
5. How to approximate?
6. First fit: Have a row of bins, insert items greedily into the first bin that fits them.
7. First fit decreasing: Sort the elements first...

Bin packing: First fit

Analysis

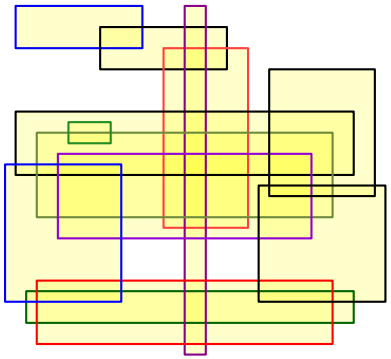
Lemma

First fit is a 2-approximation.

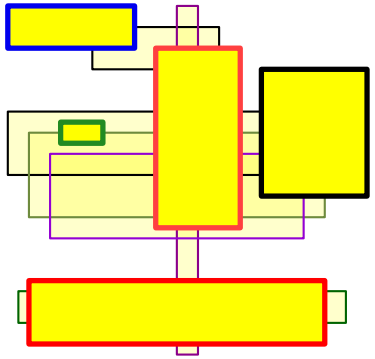
Proof.

Observe that only one bin can have less than $v/2$ content in it... □

An example



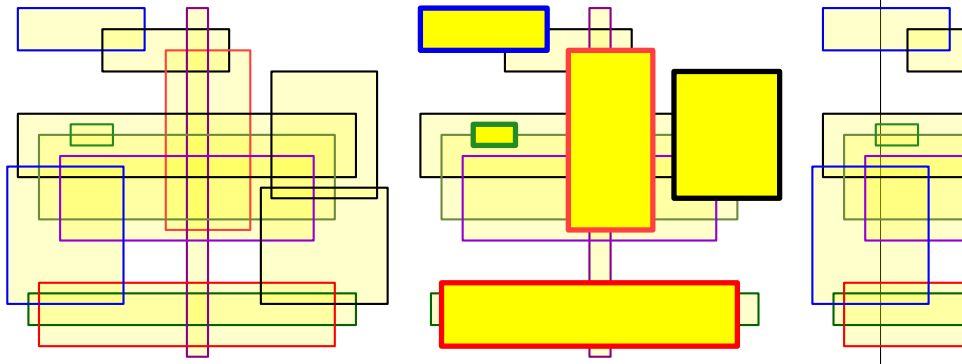
Input
Assume: Open rectangles.



Independent set of rectangles.

Independent set of rectangles

Algorithm: Divide & Conquer



Independent set of rectangles

Algorithm: Divide & Conquer

$\mathcal{R}$: A set of axis parallel rectangles.

RectIndep($\mathcal{R}$):
  if $|\mathcal{R}| \leq 10$ **then**
    Solve by brute force
    return size of solution
  x_M: Median of right x-coordinate of rects in $\mathcal{R}$
  ℓ: Vertical line through x_M.
  $\mathcal{R}_M$: Rects of $\mathcal{R}$ intersecting ℓ
  $\mathcal{R}_L$, $\mathcal{R}_R$: Rectangles in $\mathcal{R}$ left/ right of ℓ.
  $S_L \Leftarrow$ **RectIndep**($\mathcal{R}_L$)
  $S_R \Leftarrow$ **RectIndep**($\mathcal{R}_R$)
  $S_M \Leftarrow$ compute opt solution for $\mathcal{R}_M$ (intervals!)
  return $\max(S_M, S_L + S_R)$

Analysis

1. If $S_M \geq \mathrm{Opt}/(2 \lg n)$... done.
2. $\mathrm{Opt}_L + \mathrm{Opt}_R \geq (1 - 1/(2 \lg n))\mathrm{Opt}$.
3. By induction: $S_L \geq \mathrm{Opt}_L/(2 \lg(n/2))$ and $S_R \geq \mathrm{Opt}_R/(2 \lg(n/2))$.
4. $S_L + S_R \geq \frac{(1-1/(2 \lg n))\mathrm{Opt}}{2 \lg(n/2)}$
5. $$\frac{(1 - 1/(2 \lg n))}{2 \lg(n/2)} = \frac{1}{2 \lg n - 2} - \frac{1}{(2 \lg n)(2 \lg n - 2)} \geq \frac{2 \lg n - 1}{(2 \lg n)(2 \lg n - 2)} \geq \frac{2 \lg n - 2}{(2 \lg n)(2 \lg n - 2)} \geq \frac{1}{2 \lg n}.$$
6. Conclude: If $S_M \leq \mathrm{Opt}/(2 \lg n)$, then $S_L + S_R \geq \mathrm{Opt}/(2 \lg n)$.
7. Algorithm is $2 \lg n$ approximation.