NEW CS 473: Theory II, Fall 2015

Approximation Algorithms

Lecture 8

September 17, 2015

Today's Lecture

Don't give up on **NP-Hard** problems:

(A) Faster exponential time algorithms: $n^{O(n)}$, 3^n, 2^n, etc.

(B) Fixed parameter tractable.

(C) Find an approximate solution.

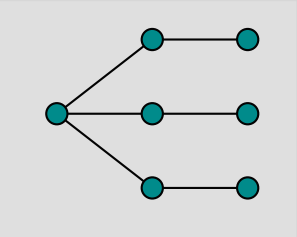
Greedy algorithms

1. **greedy algorithms**: do locally the right thing...
2. ...and they suck.

VertexCoverMin

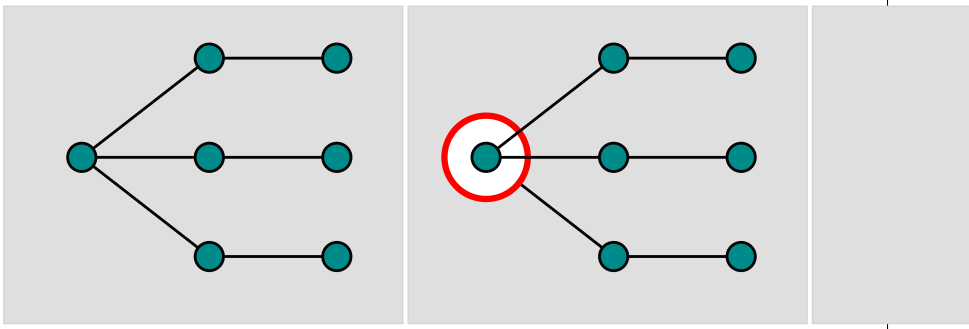
Instance: A graph $\mathbf{G}$.
Question: Return the smallest subset $S \subseteq V(\mathbf{G})$, s.t. S touches all the edges of $\mathbf{G}$.

3. **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.



Greedy algorithms

GreedyVertexCover in action...



Observation

GreedyVertexCover *returns 4 vertices, but opt is 3 vertices.*

Good enough...

Definition

In a **minimization** optimization problem, one looks for a valid solution that minimizes a certain target function.

1. VertexCoverMin: $\mathbf{Opt(G) = \min_{S \subseteq V(G), S \text{ cover of } G} |S|}$.
2. $\mathbf{VertexCover(G)}$: set realizing sol.
3. $\mathbf{Opt(G)}$: value of the target function for the optimal solution.

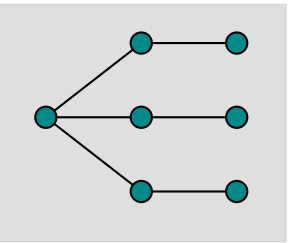
Definition

Alg is **α-approximation algorithm** for problem **Min**, achieving an approximation $\alpha \geq 1$, if for all inputs **G**, we have:

$$\frac{\mathbf{Alg(G)}}{\mathbf{Opt(G)}} \leq \alpha.$$

Back to GreedyVertexCover

1. **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.
2. Returns 4, but opt is 3!



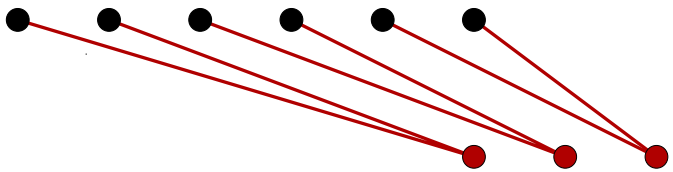
3. Can **not** be better than a $\mathbf{4/3}$-approximation algorithm.
4. Actually it is much worse!

How bad is GreedyVertexCover?

Build a bipartite graph.

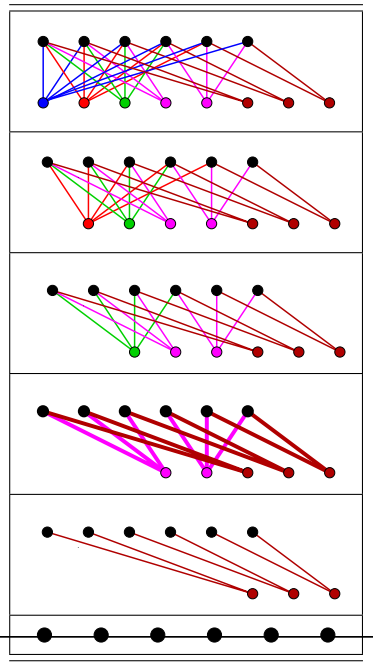
Let the top partite set be of size n.

In the bottom set add $\lfloor n/2 \rfloor$ vertices of degree **2**, such that each edge goes to a different vertex above.



Repeatedly add $\lfloor n/i \rfloor$ bottom vertices echo of degree i, for $i = 2, \ldots, n$.

How bad is GreedyVertexCover?



1. Bottom row taken by Greedy.
2. Top row was a smaller solution.

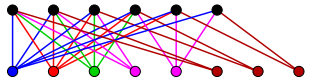
Lemma

The algorithm **GreedyVertexCover** *is* $\mathbf{\Omega(\log n)}$ *approximation to the optimal solution to VertexCoverMin.*

See notes for details!

How bad is GreedyVertexCover?

Understanding the graph...



1. Top row has n vertices.
2. Bottom row has
   - 2.1 Upper bound: $\alpha = \sum_{i=2}^{n} \lfloor n/i \rfloor \leq \sum_{i=1}^{n} n/i \leq nH_n = O(n \log n)$.
   - 2.2 Lower bound: $\alpha = \sum_{i=2}^{n} \lfloor n/i \rfloor \geq \sum_{i=1}^{n} n/i - (n-1) - n \geq n(H_n - 2) \geq n(\ln n - 2)$.
   - 2.3 Bottom row has $\Theta(n \log n)$ vertices.
3. Greedy algorithm returns bottom row $\Theta(n \log n)$ vertices.
4. Optimal solution is top row: n vertices.
5. Greedy algorithm is $O(\log n)$ approximation in this case.

Some math required

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \leq \underbrace{1 + \int_{x=1}^{n} \frac{1}{x} dx}_{1/(i+1) \leq \int_{x=i}^{i+1} (1/x) dx} = 1 + \ln n - \ln 1 = 1 + \ln n.$$

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \geq \underbrace{\int_{x=1}^{n+1} \frac{1}{x} dx}_{\frac{1}{i} \geq \int_{x=i}^{i+1} \frac{1}{x} dx} = \ln(n+1) - \ln 1 = \ln(n+1)$$

$$\geq \ln n.$$

Lemma

For $H_n = \sum_{i=1}^{n} 1/i$ we have that $\ln n \leq H_n \leq 1 + \ln n$.

Greedy Vertex Cover

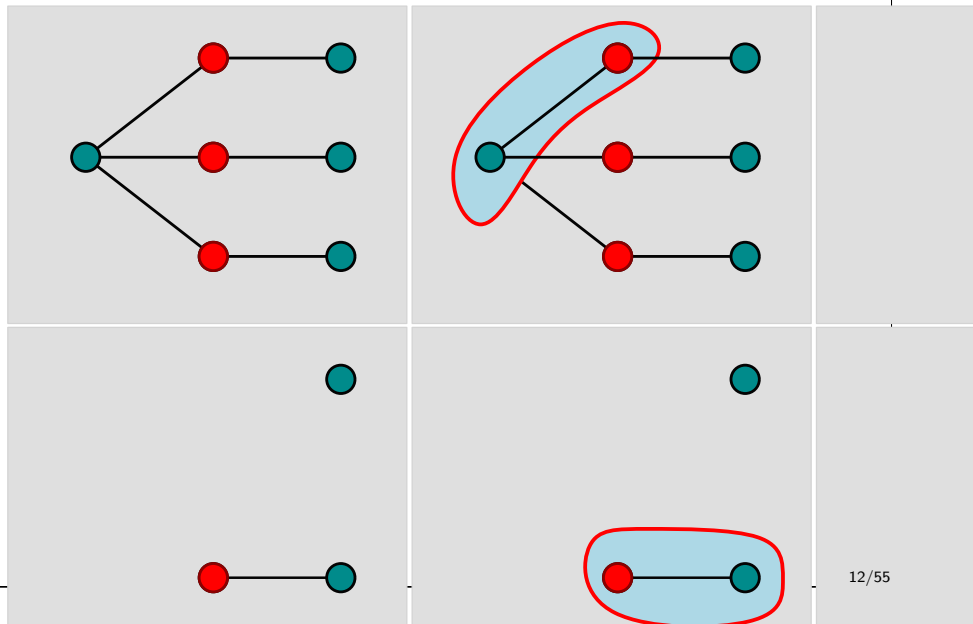
Theorem

The greedy algorithm for **VertexCover** *achieves $\Theta(\log n)$ approximation, where n (resp. m) is the number of vertices (resp., edges) in the graph. Running time is $O(mn^2)$.*

Proof

Lower bound follows from lemma.
Upper bound follows from analysis of greedy algorithm for **Set Cover**, which will be done shortly.
As for the running time, each iteration of the algorithm takes $O(mn)$ time, and there are at most n iterations.

Two for the price of one - example



Two for the price of one

```
ApproxVertexCover(G):
  S ← ∅
  while E(G) ≠ ∅ do
    uv ← any edge of G
    S ← S ∪ {u, v}
    Remove u, v from V(G)
    Remove all edges involving u or v from E(G)

  return S
```

Theorem

ApproxVertexCover *is a* **2***-approximation algorithm for VertexCoverMin that runs in* $O(n^2)$ *time.*

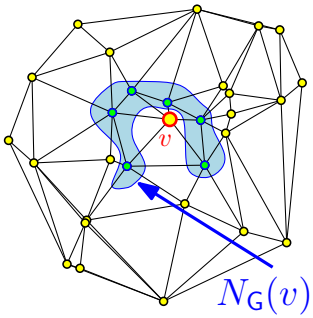
What if the vertex cover is small?

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with n vertices
2. $K \leftarrow$ Approximate VertexCoverMin up to a factor of two.
3. Any vertex cover of G is of size $\geq K/2$.
4. Naively compute optimal in $O(n^{K+2})$ time.

Neighborhood of a vertex

Definition

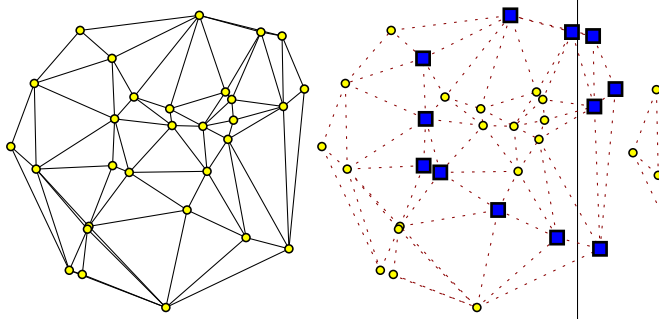
$N_{\mathsf{G}}(v)$: **Neighborhood** of v – set of vertices of $\mathbf{G}$ adjacent to v.



Induced subgraph

Definition

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph. For a subset $S \subseteq \mathbf{V}$, let $\mathbf{G}_S$ be the **induced subgraph** over S.



Exact algorithm for Set Cover

1. **G**: Input graph.
2. $\mathbf{Opt}$ = min size vertex cover for **G**. **opt** $= |\mathrm{Opt}|$.
3. Compute a set $S \subseteq \mathbf{V(G)}$ s.t. $|S| \leq \mathbf{2opt}$.
   Takes: $O(n+m)$ time.
4. Enumerate over all possible $X \subseteq S$:
   4.1 Check if X is vertex cover in **G**.
   Takes $O(n+m)$ time.
5. Return smallest VC encountered.
6. Running time:
   $O\left(2^{2\mathbf{opt}}(n+m)+n+m\right) = O(2^{2\mathbf{opt}}m)$.

Summary of result

Theorem

Given a graph **G** *with* n *vertices and* m ($\geq n$) *edges, and with a vertex cover of size* k. *Then, one can compute the optimal vertex cover in* **G** *in* $O(2^{2k}m)$ *time.*

Note, that running time is **Fixed Parameter Tractable**.

TSP

TSP-Min

Instance: **G** $= (V, E)$ a complete graph, and $\omega(e)$ a cost function on edges of **G**.
Question: The cheapest tour that visits all the vertices of **G** exactly once.

Solved exactly naively in $\approx n!$ time.
Using DP, solvable in $O(n^2 2^n)$ time.

TSP Hardness

Theorem

TSP-Min can not be approximated within **any** *factor unless* **NP = P**.

Proof.

1. Reduction from **Hamiltonian Cycle** into **TSP**.
2. **G = (V, E)**: instance of Hamiltonian cycle.
3. **J**: Complete graph over **V**.
   $$\forall u, v \in \mathbf{V} \quad w_{\mathbf{J}}(uv) = \begin{cases} 1 & uv \in \mathbf{E} \\ 2 & \text{otherwise.} \end{cases}$$
4. $\exists$ tour of price n in **J** $\iff$ $\exists$ Hamiltonian cycle in **G**.
5. No Hamiltonian cycle $\implies$ TSP price at least $n+1$.
6. But... replace **2** by cn, for c an arbitrary number

TSP Hardness - proof continued

Proof.

1. Price of all tours are either:
   (i) n (only if $\exists$ Hamiltonian cycle in $\mathbf{G}$),
   (ii) larger than $cn+1$ (actually, $\geq cn+(n-1)$).
2. Suppose you had a poly time c-approximation to TSP-Min.
3. Run it on $\mathbf{J}$:
   (i) If returned value $\geq cn+1 \implies$ no Ham Cycle since $(cn+1)/c > n$
   (ii) If returned value $\leq cn \implies$ Ham Cycle since $OPT \leq cn < cn+1$
4. c-approximation algorithm to TSP $\implies$ poly-time algorithm for **NP-Complete** problem. Possible only if $\mathbf{P = NP}$.

TSP with the triangle inequality

Because it is not that bad after all.

$\text{TSP}_{\triangle\neq}$-Min

Instance: $\mathbf{G} = (V, E)$ is a complete graph. There is also a cost function $\omega(\cdot)$ defined over the edges of $\mathbf{G}$, that complies with the triangle inequality.
Question: The cheapest tour that visits all the vertices of $\mathbf{G}$ exactly once.

triangle inequality: $\omega(\cdot)$ if

$$\forall u, v, w \in \mathbf{V(G)}, \quad \omega(u,v) \leq \omega(u,w) + \omega(w,v).$$

Shortcutting

σ: a path from s to t in $\mathbf{G} \implies \omega(st) \leq \omega(\sigma)$.

TSP with the triangle inequality

Continued...

Definition

Cycle in $\mathbf{G}$ is **Eulerian** if it visits every **edge** of $\mathbf{G}$ exactly once.

Assume you already seen the following:

Lemma

A graph $\mathbf{G}$ has a cycle that visits every edge of $\mathbf{G}$ exactly once (i.e., an Eulerian cycle) if and only if $\mathbf{G}$ is connected, and all the vertices have even degree. Such a cycle can be computed in $O(n+m)$ time, where n and m are the number of vertices and edges of $\mathbf{G}$, respectively.

TSP with the triangle inequality

Continued...

1. C_{opt} optimal **TSP** tour in $\mathbf{G}$.
2. **Observation**:
   $\omega(C_{\text{opt}}) \geq \text{weight}\Big(\text{cheapest spanning graph of } \mathbf{G}\Big)$.
3. MST: cheapest spanning graph of $\mathbf{G}$.
   $\omega(C_{\text{opt}}) \geq \omega(\text{MST}(\mathbf{G}))$
4. $O(n \log n + m) = O(n^2)$: time to compute MST.
   $n = |\mathbf{V(G)}|$, $m = \binom{n}{2}$.

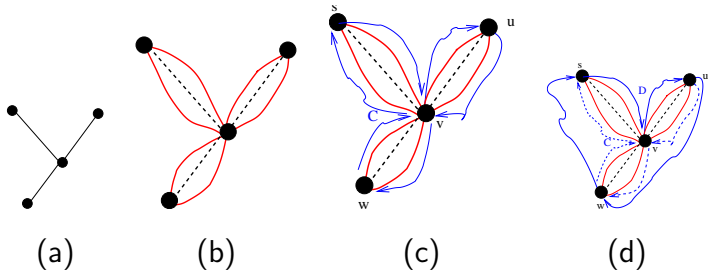
TSP with the triangle inequality

2-approximation

1. $T \leftarrow \mathrm{MST}(\mathbf{G})$
2. $\mathbf{J} \leftarrow$ duplicate very edge of T.
3. H has an Eulerian tour.
4. $\mathbf{C}$: Eulerian cycle in H.
5. $\omega(\mathbf{C}) = \omega(H) = 2\omega(T) = 2\omega(MST(\mathbf{G})) \leq 2\omega(C_{\mathrm{opt}})$.
6. π: Shortcut $\mathbf{C}$ so visit every vertex once.
7. $\omega(\pi) \leq \omega(\mathbf{C})$

TSP with the triangle inequality

2-approximation algorithm in figures



(a) (b) (c) (d)

Euler Tour: vuvwvsv
First occurrences: vuvwvsv
Shortcut String: vuwsv

TSP with the triangle inequality

2-approximation - result

Theorem

$\mathbf{G}$: *Instance of* $TSP_{\triangle\neq}$*-Min.*
C_{opt}*: min cost TSP tour of* $\mathbf{G}$.
$\Longrightarrow$ *Compute a tour of* $\mathbf{G}$ *of length* $\leq 2\omega(C_{\mathrm{opt}})$.
Running time of the algorithm is $O(n^2)$.

$\mathbf{G}$: n vertices, cost function $\omega(\cdot)$ on the edges that comply with the triangle inequality.

TSP with the triangle inequality

3/2-approximation

Definition

$\mathbf{G} = (V, E)$, a subset $M \subseteq E$ is a **matching** if no pair of edges of M share endpoints.
A **perfect matching** is a matching that covers all the vertices of $\mathbf{G}$.
w: weight function on the edges. **Min-weight perfect matching**, is the minimum weight matching among all perfect matching, where

$$\omega(M) = \sum_{e \in M} \omega(e).$$

TSP with the triangle inequality

3/2-approximation

The following is known:

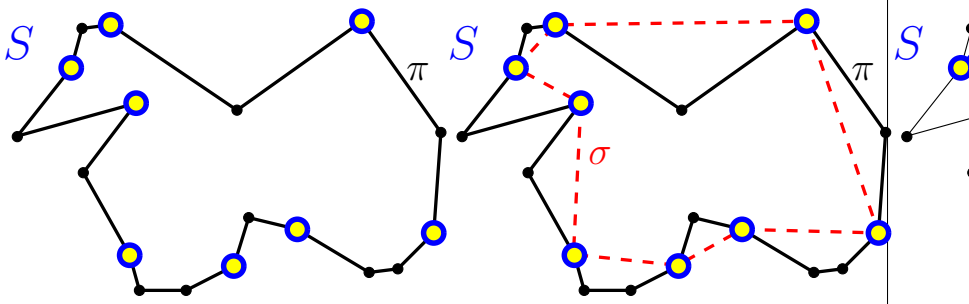
Theorem

Given a graph $\mathbf{G}$ *and weights on the edges, one can compute the min-weight perfect matching of* $\mathbf{G}$ *in polynomial time.*

Min weight perfect matching vs. TSP

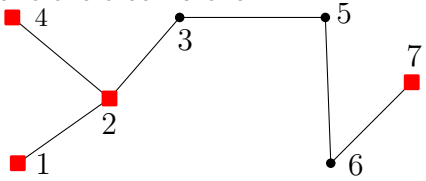
Lemma

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$*: complete graph.*
$S \subseteq \mathbf{V}$*: even size.*
$\omega(\cdot)$*: a weight function over* $\mathbf{E}$*.*
$\Longrightarrow$ *min-weight perfect matching in* $\mathbf{G}_S$ *is* $\leq \omega(\mathrm{TSP}(\mathbf{G}))/2$.



A more perfect tree?

1. How to make the tree Eulerian?



2. Pesky odd degree vertices must die!
3. Number of odd degree vertices in a graph is even!
4. Compute min-weight matching on odd vertices, and add to MST.
5. $\mathbf{J}$ = MST + (min-weight-matching) is Eulerian.
6. Weight of resulting cycle in $\mathbf{J} \leq (3/2)\omega(\mathrm{TSP})$.

Even number of odd degree vertices

Lemma

The number of odd degree vertices in any graph G' *is even.*

Proof.

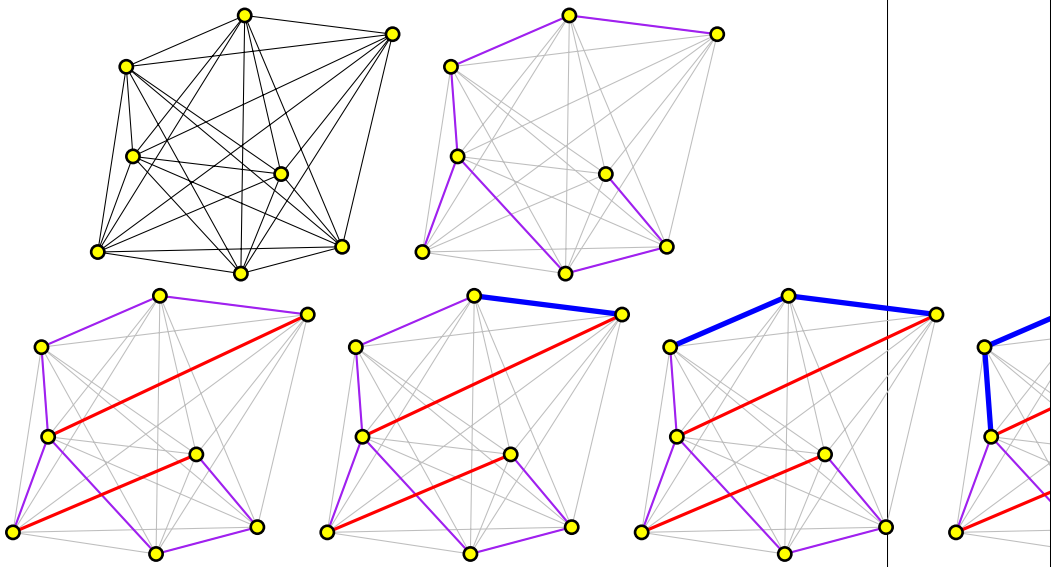
$\mu = \sum_{v \in V(G')} d(v) = 2|E(G')|$ and thus even.
$U = \sum_{v \in V(G'), d(v) \text{ is even}} d(v)$ even too.
Thus,

$$\alpha = \sum_{v \in V, d(v) \text{ is odd}} d(v) = \mu - U = \text{even number},$$

since μ and U are both even.
Number of elements in sum of all odd numbers must be even, since the total sum is even. □

3/2-approximation algorithm for TSP

Animated!

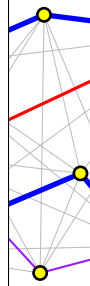


3/2-approximation algorithm for TSP

The result

Theorem

Given an instance of TSP with the triangle inequality, one can compute in polynomial time, a $(3/2)$*-approximation to the optimal TSP.*



Biographical Notes

The $3/2$-approximation for TSP with the triangle inequality is due to **Christofides [1976]**.

Exact fixed parameter tractable algorithm

Fixed parameter tractable algorithm for **VertexCoverMin**.

Computes minimum vertex cover for the induced graph $\mathbf{G}_X$:

fpVCI (X, β)

// β: size of VC computed so far.

if $X = \emptyset$ or $\mathbf{G}_X$ has no edges **then return** β

$\mathbf{e} \leftarrow$ any edge uv of $\mathbf{G}_X$.

$\beta_1 = \text{fpVCI}\Big(X \setminus \{u, v\}, \beta + 2\Big)$

$\beta_2 = \text{fpVCI}\Big(X \setminus \Big(\{u\} \cup N_{\mathbf{G}_X}(v)\Big), \beta + |N_{\mathbf{G}_X}(v)|\Big)$

$\beta_3 = \text{fpVCI}\Big(X \setminus \Big(\{v\} \cup N_{\mathbf{G}_X}(u)\Big), \beta + |N_{\mathbf{G}_X}(u)|\Big)$

return $\min(\beta_1, \beta_2, \beta_3)$.

algFPVertexCover $(\mathbf{G} = (\mathbf{V}, \mathbf{E}))$

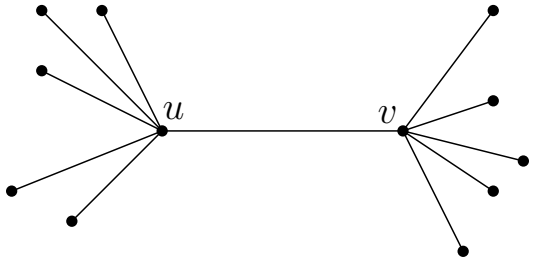
return fpVCI$(\mathbf{V}, 0)$

Depth of recursion

Lemma

The algorithm **algFPVertexCover** *returns the optimal solution to the given instance of VertexCoverMin.*

Proof...



Depth of recursion II

Lemma

The depth of the recursion of **algFPVertexCover**(**G**) *is at most* α*, where* α *is the minimum size vertex cover in* **G**.

Proof.

1. When the algorithm takes both u and v - one of them in opt. Can happen at most α times.
2. Algorithm picks $N_{G_x}(v)$ (i.e., β_2). Conceptually add v to the vertex cover being computed.
3. Do the same thing for the case of β_3.
4. Every such call add one element of the opt to conceptual set cover. Depth of recursion is $\leq \alpha$.

□

Vertex Cover

Exact fixed parameter tractable algorithm

Theorem

G: *graph with* n *vertices. Min vertex cover of size* α*. Then,* **algFPVertexCover** *returns opt. vertex cover. Running time is* $O(3^\alpha n^2)$.

Proof:

1. By lemma, recursion tree has depth α.
2. Rec-tree contains $\leq 2 \cdot 3^\alpha$ nodes.
3. Each node requires $O(n^2)$ work. ■

Algorithms with running time $O(n^c f(\alpha))$, where α is some parameter that depends on the problem are **fixed parameter tractable**.

N. Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, 1976.