5.1 NP-Completeness of Hamiltonian Cycle continued

5.1.1 Reduction from 3SAT to Hamiltonian Cycle

5.1.1.1 Directed Hamiltonian Cycle

Input: Given a directed graph $G = (V, E)$ with $n$ vertices

Goal: Does $G$ have a Hamiltonian cycle?

- A Hamiltonian cycle is a cycle in the graph that visits every vertex in $G$ exactly once

5.1.2 Reduction construction

5.1.2.1 From 3SAT to Hamiltonian cycle in directed graph

(A) Given 3SAT formula $\varphi$ create a graph $G_\varphi$ such that

- $G_\varphi$ has a Hamiltonian cycle if and only if $\varphi$ is satisfiable
- $G_\varphi$ should be constructible from $\varphi$ by a polynomial time algorithm $A$

(B) Notation: $\varphi$ has $n$ variables $x_1, x_2, \ldots, x_n$ and $m$ clauses $C_1, C_2, \ldots, C_m$. 
5.1.3 The Reduction: By figure

5.1.3.1 More details were given in the previous lecture

3SAT formula $\varphi$:

$$\varphi = (x_1 \lor \neg x_2 \lor x_4)$$
$$\land (\neg x_1 \lor \neg x_2 \lor \neg x_3)$$

5.1.3.2 The Reduction: Assignment $\Rightarrow$ Hamiltonian cycle

3SAT formula $\varphi$:

$$\varphi = (x_1 \lor \neg x_2 \lor x_4)$$
$$\land (\neg x_1 \lor \neg x_2 \lor \neg x_3)$$

A satisfying assignment:

$x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$
5.1.3.3 Reduction: Assignment ⇒ Hamiltonian cycle
Satisfying assignment: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$

Conclude: If $\varphi$ has a satisfying assignment then there is an Hamiltonian cycle in $G_\varphi$.

5.1.4 Reduction: Hamiltonian cycle $\Rightarrow$ Assignment

5.1.4.1 No shenanigan: Hamiltonian cycle can not leave a row in the middle
Conclude: Hamiltonian cycle must go through each row completely from left to right, or right to left. As such, can be interpreted as a valid assignment.

5.1.5 Reduction: Hamiltonian cycle ⇒ Assignment

5.1.5.1 Drawing example

5.1.5.2 Correctness Proof

Proposition $\varphi$ has a satisfying assignment iff $G_\varphi$ has a Hamiltonian cycle.

Proof: $\Rightarrow$ Let $a$ be the satisfying assignment for $\varphi$. Define Hamiltonian cycle as follows

- If $a(x_i) = 1$ then traverse path $i$ from left to right
– If \( a(x_i) = 0 \) then traverse path \( i \) from right to left
– For each clause, path of at least one variable is in the “right” direction to splice in the node corresponding to clause

5.1.5.3 Hamiltonian Cycle ⇒ Satisfying assignment

Suppose \( \Pi \) is a Hamiltonian cycle in \( G_\phi \)

• If \( \Pi \) enters \( c_j \) (vertex for clause \( C_j \)) from vertex \( 3j \) on path \( i \) then it must leave the clause vertex on edge to \( 3j + 1 \) on the same path \( i \)
  – If not, then only unvisited neighbor of \( 3j + 1 \) on path \( i \) is \( 3j + 2 \)
  – Thus, we don’t have two unvisited neighbors (one to enter from, and the other to leave) to have a Hamiltonian Cycle

• Similarly, if \( \Pi \) enters \( c_j \) from vertex \( 3j + 1 \) on path \( i \) then it must leave the clause vertex \( c_j \) on edge to \( 3j \) on path \( i \)

5.1.5.4 Hamiltonian Cycle ⇒ Satisfying assignment (contd)

• Thus, vertices visited immediately before and after \( C_i \) are connected by an edge

• We can remove \( c_j \) from cycle, and get Hamiltonian cycle in \( G - c_j \)

• Consider Hamiltonian cycle in \( G - \{c_1, \ldots, c_m\} \); it traverses each path in only one direction, which determines the truth assignment

5.2 Hamiltonian cycle in undirected graph

5.2.0.1 Hamiltonian Cycle

Problem 5.2.1. Input Given undirected graph \( G = (V, E) \)

Goal Does \( G \) have a Hamiltonian cycle? That is, is there a cycle that visits every vertex exactly one (except start and end vertex)?

5.2.0.2 NP-Completeness

Theorem 5.2.2. Hamiltonian cycle problem for undirected graphs is NP-Complete.

Proof: • The problem is in NP; proof left as exercise.
  • Hardness proved by reducing Directed Hamiltonian Cycle to this problem
5.2.0.3 Reduction Sketch

**Goal:** Given directed graph $G$, need to construct undirected graph $G'$ such that $G$ has Hamiltonian Path iff $G'$ has Hamiltonian path Reduction

- Replace each vertex $v$ by 3 vertices: $v_{in}$, $v$, and $v_{out}$
- A directed edge $(a, b)$ is replaced by edge $(a_{out}, b_{in})$

5.2.1 Hamiltonian cycle reduction

5.2.1.1 Undirected to directed case

5.2.1.2 Reduction: Wrapup

- The reduction is polynomial time (exercise)
- The reduction is correct (exercise)
5.3 NP-Completeness of Graph Coloring

5.3.0.1 Graph Coloring

**Graph Coloring**

**Instance:** $G = (V, E)$: Undirected graph, integer $k$.

**Question:** Can the vertices of the graph be colored using $k$ colors so that vertices connected by an edge do not get the same color?

5.3.0.2 Graph 3-Coloring

**3 Coloring**

**Instance:** $G = (V, E)$: Undirected graph.

**Question:** Can the vertices of the graph be colored using 3 colors so that vertices connected by an edge do not get the same color?

5.3.0.3 Graph Coloring

(A) **Observation:** If $G$ is colored with $k$ colors then each color class (nodes of same color) form an independent set in $G$.

(B) $G$ can be partitioned into $k$ independent sets iff $G$ is $k$-colorable.

(C) Graph 2-Coloring can be decided in polynomial time.

(D) $G$ is 2-colorable iff $G$ is bipartite!

(E) There is a linear time algorithm to check if $G$ is bipartite using BFS (we saw this earlier).

5.3.1 Problems related to graph coloring

5.3.1.1 Graph Coloring and Register Allocation

Register Allocation Assign variables to (at most) $k$ registers such that variables needed at the same time are not assigned to the same register. Interference Graph Vertices are variables, and there is an edge between two vertices, if the two variables are “live” at the same time. Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with $k$ colors

- Moreover, $3\text{-COLOR} \leq_p k\text{-Register Allocation}$, for any $k \geq 3$

5.3.1.2 Class Room Scheduling

(A) Given $n$ classes and their meeting times, are $k$ rooms sufficient?

(B) Reduce to Graph $k$-Coloring problem

(C) Create graph $G$

- a node $v_i$ for each class $i$
- an edge between $v_i$ and $v_j$ if classes $i$ and $j$ conflict

(D) Exercise: $G$ is $k$-colorable iff $k$ rooms are sufficient.
5.3.1.3 Frequency Assignments in Cellular Networks

(A) Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)

- Breakup a frequency range \([a, b]\) into disjoint bands of frequencies \([a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]\)
- Each cell phone tower (simplifying) gets one band
- Constraint: nearby towers cannot be assigned same band, otherwise signals will interfere

(B) Problem: given \(k\) bands and some region with \(n\) towers, is there a way to assign the bands to avoid interference?

(C) Can reduce to \(k\)-coloring by creating intereference/conflict graph on towers.

5.4 Showing hardness of 3 COLORING

5.4.0.1 3-Coloring is NP-Complete

- **3-Coloring** is in \(\text{NP}\).
  - **Certificate**: for each node a color from \(\{1, 2, 3\}\).
  - **Certifier**: Check if for each edge \((u, v)\), the color of \(u\) is different from that of \(v\).

- **Hardness**: We will show \(3\text{-SAT} \leq_P 3\text{-Coloring}\).

5.4.0.2 Reduction Idea

(A) \(\varphi\): Given \(3\text{SAT}\) formula (i.e., \(3\text{CNF}\) formula).

(B) \(\varphi\): variables \(x_1, \ldots, x_n\) and clauses \(C_1, \ldots, C_m\).

(C) Create graph \(G_\varphi\) s.t. \(G_\varphi\) 3-colorable \(\iff\) \(\varphi\) satisfiable.

- \(i\rightarrow i\): encode assignment \(x_1, \ldots, x_n\) in colors assigned nodes of \(G_\varphi\).
- \(i\rightarrow i\): create triangle with node True, False, Base
- \(i\rightarrow i\): for each variable \(x_i\) two nodes \(v_i\) and \(\bar{v}_i\) connected in a triangle with common Base
- \(i\rightarrow i\): If graph is 3-colored, either \(v_i\) or \(\bar{v}_i\) gets the same color as True. Interpret this as a truth assignment to \(v_i\)
- \(i\rightarrow i\): Need to add constraints to ensure clauses are satisfied (next phase)
5.4.0.4 Clause Satisfiability Gadget

(A) For each clause \( C_j = (a \lor b \lor c) \), create a small gadget graph

- gadget graph connects to nodes corresponding to \( a, b, c \)
- needs to implement OR

(B) OR-gadget-graph:

5.4.0.5 OR-Gadget Graph

**Property:** if \( a, b, c \) are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

**Property:** if one of \( a, b, c \) is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

5.4.0.6 Reduction

- create triangle with nodes True, False, Base

- for each variable \( x_i \), two nodes \( v_i \) and \( \bar{v}_i \) connected in a triangle with common Base

- for each clause \( C_j = (a \lor b \lor c) \), add OR-gadget graph with input nodes \( a, b, c \) and connect output node of gadget to both False and Base
5.4.0.7 Reduction

Claim 5.4.1. No legal 3-coloring of above graph (with coloring of nodes \( T, F, B \) fixed) in which \( a, b, c \) are colored False. If any of \( a, b, c \) are colored True then there is a legal 3-coloring of above graph.

5.4.0.8 3 coloring of the clause gadget

5.4.0.9 Reduction Outline

Example 5.4.2. \( \varphi = (u \lor \lnot v \lor w) \land (v \lor x \lor \lnot y) \)

5.4.0.10 Correctness of Reduction

\( \varphi \) is satisfiable implies \( G_\varphi \) is 3-colorable

\( i \to i^+ \) if \( x_i \) is assigned True, color \( v_i \) True and \( \bar{v}_i \) False

\( i \to i^- \) for each clause \( C_j = (a \lor b \lor c) \) at least one of \( a, b, c \) is colored True. OR-gadget for \( C_j \) can be 3-colored such that output is True.
Gφ is 3-colorable implies φ is satisfiable

¡+-¡. if v_i is colored True then set x_i to be True, this is a legal truth assignment

¡+-¡. consider any clause C_j = (a ∨ b ∨ c). it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

5.4.1 Graph generated in reduction...

5.4.1.1 ... from 3SAT to 3COLOR

(a ∨ b ∨ c) ∧ (b ∨ c ∨ d) ∧ (d ∨ c ∨ d) ∧ (a ∨ b ∨ d)
5.5 Hardness of Subset Sum

5.5.0.1 Subset Sum

**Subset Sum**

**Instance:** \( S \) - set of positive integers, \( t \): - an integer number (Target)

**Question:** Is there a subset \( X \subseteq S \) such that \( \sum_{x \in X} x = t \)?

Claim 5.5.1. **Subset Sum** is **NP-Complete**.

5.5.0.2 Vec Subset Sum

We will prove following problem is **NP-Complete**...

**Vec Subset Sum**

**Instance:** \( S \) - set of \( n \) vectors of dimension \( k \), each vector has non-negative numbers for its coordinates, and a target vector \( \vec{t} \).

**Question:** Is there a subset \( X \subseteq S \) such that \( \sum_{\vec{x} \in X} \vec{x} = \vec{t} \)?

Reduction from **3SAT**.

5.5.1 Vec Subset Sum

5.5.1.1 Handling a single clause

Think about vectors as being lines in a table.
First gadget

Selecting between two lines.

<table>
<thead>
<tr>
<th>Target</th>
<th>??</th>
<th>??</th>
<th>01</th>
<th>??</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>??</td>
<td>??</td>
<td>01</td>
<td>??</td>
</tr>
<tr>
<td>$a_2$</td>
<td>??</td>
<td>??</td>
<td>01</td>
<td>??</td>
</tr>
</tbody>
</table>

Two rows for every variable $x$: selecting either $x = 0$ or $x = 1$.

5.5.1.2 Handling a clause...

We will have a column for every clause...

<table>
<thead>
<tr>
<th>numbers</th>
<th>...</th>
<th>$C \equiv a \lor b \lor \overline{c}$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>...</td>
<td>01</td>
<td>...</td>
</tr>
<tr>
<td>$\overline{a}$</td>
<td>...</td>
<td>00</td>
<td>...</td>
</tr>
<tr>
<td>$b$</td>
<td>...</td>
<td>01</td>
<td>...</td>
</tr>
<tr>
<td>$\overline{b}$</td>
<td>...</td>
<td>00</td>
<td>...</td>
</tr>
<tr>
<td>$c$</td>
<td>...</td>
<td>00</td>
<td>...</td>
</tr>
<tr>
<td>$\overline{c}$</td>
<td>...</td>
<td>01</td>
<td>...</td>
</tr>
<tr>
<td>$C$ fix-up 1</td>
<td>000</td>
<td>07</td>
<td>000</td>
</tr>
<tr>
<td>$C$ fix-up 2</td>
<td>000</td>
<td>08</td>
<td>000</td>
</tr>
<tr>
<td>$C$ fix-up 3</td>
<td>000</td>
<td>09</td>
<td>000</td>
</tr>
<tr>
<td>TARGET</td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.5.1.3 3SAT to Vec Subset Sum

<table>
<thead>
<tr>
<th>numbers</th>
<th>$a \lor \pi$</th>
<th>$b \lor \pi$</th>
<th>$c \lor \pi$</th>
<th>$d \lor \pi$</th>
<th>$D \equiv \overline{e} \lor b \lor \overline{c}$</th>
<th>$C \equiv a \lor b \lor \pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>$\overline{a}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>00</td>
<td>00</td>
</tr>
<tr>
<td>$b$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>$\overline{b}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>01</td>
<td>00</td>
</tr>
<tr>
<td>$c$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>$\overline{c}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>01</td>
<td>00</td>
</tr>
<tr>
<td>$D$ fix-up 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>01</td>
<td>01</td>
</tr>
<tr>
<td>$D$ fix-up 2</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>01</td>
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</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>01</td>
<td>00</td>
</tr>
<tr>
<td>$TARGET$</td>
<td>10</td>
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<td>10</td>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.5.1.4 Vec Subset Sum to Subset Sum

<table>
<thead>
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<th>numbers</th>
</tr>
</thead>
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<tr>
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</tr>
<tr>
<td>000000000900</td>
</tr>
<tr>
<td>0010101011010</td>
</tr>
</tbody>
</table>

5.5.1.5 Other NP-Complete Problems

- 3-Dimensional Matching
- Subset Sum

Read book.

5.5.1.6 Need to Know NP-Complete Problems

- 3SAT.
- Circuit-SAT.
- Independent Set.
- Vertex Cover.
- Clique.
- Set Cover / Hitting Set.
- Hamiltonian Cycle (in Directed/Undirected Graphs).
- 3Coloring.
- 3-D Matching.
- Subset Sum / Partition.
5.5.1.7 Subset Sum and Knapsack

(A) **Subset Sum Problem:** Given \(n\) integers \(a_1, a_2, \ldots, a_n\) and a target \(B\), is there a subset of \(S\) of \(\{a_1, \ldots, a_n\}\) such that the numbers in \(S\) add up *precisely* to \(B\)?

(B) Subset Sum is **NP-Complete**— see book.

(C) **Knapsack:** Given \(n\) items with item \(i\) having size \(s_i\) and profit \(p_i\), a knapsack of capacity \(B\), and a target profit \(P\), is there a subset \(S\) of items that can be packed in the knapsack and the profit of \(S\) is at least \(P\)?

(D) Show Knapsack problem is **NP-Complete** via reduction from Subset Sum (exercise).

5.5.1.8 Subset Sum and Knapsack

(A) Subset Sum can be solved in \(O(nB)\) time using dynamic programming (exercise).

(B) Implies that problem is hard only when numbers \(a_1, a_2, \ldots, a_n\) are exponentially large compared to \(n\). That is, each \(a_i\) requires polynomial in \(n\) bits.

(C) *Number problems* of the above type are said to be **weakly NPComplete**.