Greedy Algorithms for Minimum Spanning Trees

Lecture 12
October 9, 2014
Part I

Greedy Algorithms: Minimum Spanning Tree
**Minimum Spanning Tree**

**Input**  Connected graph \( G = (V, E) \) with edge costs

**Goal**  Find \( T \subseteq E \) such that \( (V, T) \) is connected and total cost of all edges in \( T \) is smallest

1. \( T \) is the minimum spanning tree (MST) of \( G \)
Minimum Spanning Tree

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Applications

1. Network Design
   1. Designing networks with minimum cost but maximum connectivity

2. Approximation algorithms
   1. Can be used to bound the optimality of algorithms to approximate Traveling Salesman Problem, Steiner Trees, etc.

3. Cluster Analysis
Greedy Template

Initially $E$ is the set of all edges in $G$

$T$ is empty (* $T$ will store edges of a MST *)

while $E$ is not empty do

    choose $i \in E$

    if ($i$ satisfies condition)
        add $i$ to $T$

return the set $T$

Main Task: In what order should edges be processed? When should we add edge to spanning tree?
Kruskal’s Algorithm

Process edges in the order of their costs (starting from the least) and add edges to $T$ as long as they don’t form a cycle.

Figure: Graph $G$

Figure: MST of $G$
Kruskal’s Algorithm

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Prim’s Algorithm

$T$ maintained by algorithm will be a tree. Start with a node in $T$. In each iteration, pick edge with least attachment cost to $T$.

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Figure: MST of $G$
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![Graph G](image1)

![MST of G](image2)

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**Figure :** Graph $G$

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Prim’s Algorithm

$T$ maintained by algorithm will be a tree. Start with a node in $T$. In each iteration, pick edge with least attachment cost to $T$.

Figure: Graph $G$

Figure: MST of $G$
Prim’s Algorithm

T maintained by algorithm will be a tree. Start with a node in T. In each iteration, pick edge with least attachment cost to T.

Figure: Graph G

Figure: MST of G
Reverse Delete Algorithm

Initially \( E \) is the set of all edges in \( G \). \( T \) is \( E \) (* \( T \) will store edges of a MST *)

while \( E \) is not empty do

choose \( i \in E \) of largest cost

if removing \( i \) does not disconnect \( T \) then

remove \( i \) from \( T \)

end if

end while

return the set \( T \)

Returns a minimum spanning tree.
Can we use Prim’s algorithm for MST to find the Shortest Path?

(A) Yes. Prim’s algorithm uses the same principle as Dijkstra.

(B) No. Shortest path is NP-hard and Prim runs in polynomial time.

(C) No. Shortest path algorithms like Dijkstra, preserve a global optimality invariant, whereas MST can be found with non-adaptive greedy choices.

(D) IDK.
Many different MST algorithms

All of them rely on some basic properties of MSTs, in particular the Cut Property to be seen shortly.
Assumption

And for now . . .

Assumption

Edge costs are distinct, that is no two edge costs are equal.
Cuts

Definition

Given a graph $G = (V, E)$, a cut is a partition of the vertices of the graph into two sets $(S, V \setminus S)$.

Edges having an endpoint on both sides are the edges of the cut.

A cut edge is crossing the cut.
Cuts

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Given a graph $G = (V, E)$, a cut is a partition of the vertices of the graph into two sets $(S, V \setminus S)$.

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A cut edge is crossing the cut.
Safe and Unsafe Edges

**Definition**

An edge $e = (u, v)$ is a **safe** edge if there is some partition of $V$ into $S$ and $V \setminus S$ and $e$ is the unique minimum cost edge crossing $S$ (one end in $S$ and the other in $V \setminus S$).

**Definition**

An edge $e = (u, v)$ is an **unsafe** edge if there is some cycle $C$ such that $e$ is the unique maximum cost edge in $C$.

**Proposition**

*If edge costs are distinct then every edge is either safe or unsafe.*

**Proof.**

*Exercise.*
**Definition**

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If edge costs are distinct then every edge is either safe or unsafe.

Proof.
Exercise.
Every cut identifies one safe edge...

...the cheapest edge in the cut.

Note: An edge $e$ may be a safe edge for many cuts!
Safe edge

Example...

Every cut identifies one safe edge...

...the cheapest edge in the cut.

Note: An edge \( e \) may be a safe edge for many cuts!
Every cycle identifies one **unsafe** edge...

...the most expensive edge in the cycle.
Every cycle identifies one **unsafe** edge...

...the most expensive edge in the cycle.
Figure: Graph with unique edge costs. Safe edges are red, rest are unsafe.

And all safe edges are in the MST in this case...
Figure: Graph with unique edge costs. Safe edges are red, rest are unsafe.

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Figure: Graph with unique edge costs. Safe edges are red, rest are unsafe.

And all safe edges are in the **MST** in this case...
Key Observation: Cut Property

**Lemma**

*If e is a safe edge then every minimum spanning tree contains e.*

**Proof.**

1. Suppose (for contradiction) e is not in MST T.
2. Since e is safe there is an S ⊂ V such that e is the unique min cost edge crossing S.
3. Since T is connected, there must be some edge f with one end in S and the other in V \ S.
4. Since c_f > c_e, T' = (T \ {f}) ∪ {e} is a spanning tree of lower cost! Error: T' may not be a spanning tree!!
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*If* e *is a safe edge then every minimum spanning tree contains* e.

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Problematic example. \( S = \{1, 2, 7\}, \ e = (7, 3), \ f = (1, 6) \). \( T - f + e \) is not a spanning tree.

(A) Consider adding the edge \( f \).
Problematic example. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$. $T - f + e$ is not a spanning tree.

1. (A) Consider adding the edge $f$.
2. (B) It is safe because it is the cheapest edge in the cut.
Problematic example. \( S = \{1, 2, 7\}, \ e = (7, 3), \ f = (1, 6). \ T - f + e \) is not a spanning tree.

(A) Consider adding the edge \( f \).
(B) It is safe because it is the cheapest edge in the cut.
(C) Let's throw out the edge \( e \) currently in the spanning tree which is more expensive than \( f \) and is in the same cut. Put it \( f \) instead...
Error in Proof: Example

Problematic example. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$. $T - f + e$ is not a spanning tree.

1. (A) Consider adding the edge $f$.
2. (B) It is safe because it is the cheapest edge in the cut.
3. (C) Let's throw out the edge $e$ currently in the spanning tree which is more expensive than $f$ and is in the same cut. Put it $f$ instead...
4. (D) New graph of selected edges is not a tree anymore. BUG.
The **min-cut** of a graph $G$ is a partition of $G$ in $(S, V \setminus S)$ that minimizes the number of edges that cross the cut, $E(S, V \setminus S)$. The **sparsest-cut** of $G$ is a partition $(S, V \setminus S)$ that minimizes the ratio $\phi(G) = \frac{E(S, V \setminus S)}{|S||V\setminus S|}$. Is the *min-cut* achieved by the same partition as the *sparsest-cut*?

(A) Yes. The ratio $\phi(G)$ is minimized when $E(S, V \setminus S)$ is minimized.

(B) No. Mincut is in P but sparsest-cut is NP-complete.

(C) Yes. They can both be solved by a greedy algorithm.

(D) No. sparsest-cut is in P but min-cut is NP-Complete.
Proof of Cut Property

Proof.

1. Suppose \( e = (v, w) \) is not in \( \text{MST}_T \) and \( e \) is min weight edge in cut \( (S, V \setminus S) \). Assume \( v \in S \).

2. \( T \) is spanning tree: there is a unique path \( P \) from \( v \) to \( w \) in \( T \).

3. Let \( w' \) be the first vertex in \( P \) belonging to \( V \setminus S \); let \( v' \) be the vertex just before it on \( P \), and let \( e' = (v', w') \).

4. \( T' = (T \setminus \{e'\}) \cup \{e\} \) is spanning tree of lower cost. (Why?)
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Proof of Cut Property (contd)

Observation

\[ T' = (T \setminus \{e'\}) \cup \{e\} \] is a spanning tree.

Proof.

\( T' \) is connected.

Removal of edge \( e' = (v', w') \) from \( T \) but \( v' \) and \( w' \) are connected by the path \( P - f + e \) in \( T' \). Hence \( T' \) is connected if \( T \) is.

\( T' \) is a tree

\( T' \) is connected and has \( n - 1 \) edges (since \( T \) had \( n - 1 \) edges) and hence \( T' \) is a tree.
Proof of Cut Property (contd)

**Observation**

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**Proof.**

- **T'** is connected.
  - Removed \( e' = (v', w') \) from \( T \) but \( v' \) and \( w' \) are connected by the path \( P = f + e \) in \( T' \). Hence \( T' \) is connected if \( T \) is.
- **T'** is a tree
  - \( T' \) is connected and has \( n - 1 \) edges (since \( T \) had \( n - 1 \) edges) and hence \( T' \) is a tree.
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\( T' \) is connected and has \( n - 1 \) edges (since \( T \) had \( n - 1 \) edges) and hence \( T' \) is a tree.
Lemma

Let $G$ be a connected graph with distinct edge costs, then the set of safe edges form a connected graph.

Proof.

1. Suppose not. Let $S$ be a connected component in the graph induced by the safe edges.

2. Consider the edges crossing $S$, there must be a safe edge among them since edge costs are distinct and so we must have picked it.
Corollary

Let $G$ be a connected graph with distinct edge costs, then set of safe edges form the unique MST of $G$.

Consequence: Every correct MST algorithm when $G$ has unique edge costs includes exactly the safe edges.
Safe Edges form an MST

**Corollary**

Let $G$ be a connected graph with distinct edge costs, then set of safe edges form the *unique* MST of $G$.

**Consequence:** Every correct MST algorithm when $G$ has unique edge costs includes exactly the safe edges.
Lemma

If $e$ is an unsafe edge then no \textit{MST} of $G$ contains $e$.

Proof.

Exercise. See textbook.

Note: Cut and Cycle properties hold even when edge costs are not distinct. Safe and unsafe definitions do not rely on distinct cost assumption.
Correctness of Prim’s Algorithm

**Prim’s Algorithm**

Pick edge with minimum attachment cost to current tree, and add to current tree.

**Proof of correctness.**

1. **If** $e$ **is added to tree, then** $e$ **is safe and belongs to every MST.**
   - Let $S$ be the vertices connected by edges in $T$ when $e$ is added.
   - $e$ is edge of lowest cost with one end in $S$ and the other in $V \setminus S$ and hence $e$ is safe.

2. **Set of edges output is a spanning tree**
   - Set of edges output forms a connected graph: by induction, $S$ is connected in each iteration and eventually $S = V$.
   - Only safe edges added and they do not have a cycle.
Correctness of Prim’s Algorithm

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Proof of correctness.
1. If e is added to tree, then e is safe and belongs to every MST.
   1. Let S be the vertices connected by edges in T when e is added.
   2. e is edge of lowest cost with one end in S and the other in V \ S and hence e is safe.
2. Set of edges output is a spanning tree
   1. Set of edges output forms a connected graph: by induction, S is connected in each iteration and eventually S = V.
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   1. Set of edges output forms a connected graph: by induction, \( S \) is connected in each iteration and eventually \( S = V \).
   2. Only safe edges added and they do not have a cycle
Correctness of Kruskal’s Algorithm

Kruskal’s Algorithm

Pick edge of lowest cost and add if it does not form a cycle with existing edges.

Proof of correctness.

1. If \( e = (u, v) \) is added to tree, then \( e \) is safe
   1. When algorithm adds \( e \) let \( S \) and \( S' \) be the connected components containing \( u \) and \( v \) respectively
   2. \( e \) is the lowest cost edge crossing \( S \) (and also \( S' \)).
   3. If there is an edge \( e' \) crossing \( S \) and has lower cost than \( e \), then \( e' \) would come before \( e \) in the sorted order and would be added by the algorithm to \( T \)

2. Set of edges output is a spanning tree: exercise
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Kruskal’s Algorithm
Pick edge of lowest cost and add if it does not form a cycle with existing edges.

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2. Set of edges output is a spanning tree : exercise
Correctness of Kruskal’s Algorithm

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Pick edge of lowest cost and add if it does not form a cycle with existing edges.

Proof of correctness.

1. If $e = (u, v)$ is added to tree, then $e$ is safe
   1. When algorithm adds $e$ let $S$ and $S'$ be the connected components containing $u$ and $v$ respectively
   2. $e$ is the lowest cost edge crossing $S$ (and also $S'$).
   3. If there is an edge $e'$ crossing $S$ and has lower cost than $e$, then $e'$ would come before $e$ in the sorted order and would be added by the algorithm to $T$

2. Set of edges output is a spanning tree: exercise
Correctness of Reverse Delete Algorithm

Reverse Delete Algorithm
Consider edges in decreasing cost and remove an edge if it does not disconnect the graph

Proof of correctness.
Argue that only unsafe edges are removed (see textbook).
What does MST stand for...

...according to popular culture? (google)

(A) Masters of Sacred Theology.
(B) Minimum Spanning Tree.
(C) Missouri University of Science and Technology.
(D) Mountain Standard Time.
When edge costs are not distinct

Heuristic argument: Make edge costs distinct by adding a small tiny and different cost to each edge

Formal argument: Order edges lexicographically to break ties

1. $e_i \prec e_j$ if either $c(e_i) < c(e_j)$ or $(c(e_i) = c(e_j)$ and $i < j)$

2. Lexicographic ordering extends to sets of edges. If $A, B \subseteq E$, $A \neq B$ then $A \prec B$ if either $c(A) < c(B)$ or $(c(A) = c(B)$ and $A \setminus B$ has a lower indexed edge than $B \setminus A$)

3. Can order all spanning trees according to lexicographic order of their edge sets. Hence there is a unique MST.

Prim’s, Kruskal, and Reverse Delete Algorithms are optimal with respect to lexicographic ordering.
When edge costs are not distinct

**Heuristic argument:** Make edge costs distinct by adding a small tiny and different cost to each edge.

**Formal argument:** Order edges lexicographically to break ties.

1. $e_i \prec e_j$ if either $c(e_i) < c(e_j)$ or $(c(e_i) = c(e_j)$ and $i < j)$

2. Lexicographic ordering extends to sets of edges. If $A, B \subseteq E$, $A \neq B$ then $A \prec B$ if either $c(A) < c(B)$ or $(c(A) = c(B)$ and $A \setminus B$ has a lower indexed edge than $B \setminus A$)

3. Can order all spanning trees according to lexicographic order of their edge sets. Hence there is a unique **MST**.

Prim’s, Kruskal, and Reverse Delete Algorithms are optimal with respect to lexicographic ordering.
When edge costs are not distinct

Heuristic argument: Make edge costs distinct by adding a small tiny and different cost to each edge

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3. Can order all spanning trees according to lexicographic order of their edge sets. Hence there is a unique MST.

Prim’s, Kruskal, and Reverse Delete Algorithms are optimal with respect to lexicographic ordering.
Edge Costs: Positive and Negative

1. Algorithms and proofs don’t assume that edge costs are non-negative! MST algorithms work for arbitrary edge costs.

2. Another way to see this: make edge costs non-negative by adding to each edge a large enough positive number. Why does this work for MSTs but not for shortest paths?

3. Can compute maximum weight spanning tree by negating edge costs and then computing an MST.
Part II

Data Structures for MST: Priority Queues and Union-Find
Implementing Prim’s Algorithm

Prim_ComputeMST

\[
\begin{align*}
E \text{ is the set of all edges in } G \\
S &= \{1\} \\
T \text{ is empty } (* T \text{ will store edges of a MST } *) \\
\text{while } S \neq V \text{ do} \\
&\quad \text{pick } e = (v, w) \in E \text{ such that} \\
&\quad \quad v \in S \text{ and } w \in V - S \\
&\quad \quad e \text{ has minimum cost} \\
&\quad T = T \cup e \\
&\quad S = S \cup w \\
\text{return the set } T
\end{align*}
\]

Analysis

1. Number of iterations = \( O(n) \), where \( n \) is number of vertices
2. Picking \( e \) is \( O(m) \), where \( m \) is the number of edges
3. Total time \( O(nm) \)
Implementing Prim’s Algorithm

Prim\_ComputeMST

- \( E \) is the set of all edges in \( G \)
- \( S = \{1\} \)
- \( T \) is empty (\( T \) will store edges of a MST *)

while \( S \neq V \) do
  pick \( e = (v, w) \in E \) such that
  - \( v \in S \) and \( w \in V - S \)
  - \( e \) has minimum cost
  \( T = T \cup e \)
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return the set \( T \)

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  - \(T = T \cup e\)
  - \(S = S \cup w\)

return the set **T**

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Implementing Prim’s Algorithm
More Efficient Implementation

Prim.ComputeMST

E is the set of all edges in G
S = {1}
T is empty (* T will store edges of a MST *)
for v \not\in S, a(v) = \min_{w \in S} c(w, v)
for v \not\in S, e(v) = w such that w \in S and c(w, v) is minimum
while S \neq V do
    pick v with minimum a(v)
    T = T \cup \{(e(v), v)\}
    S = S \cup \{v\}
    update arrays a and e
return the set T

Maintain vertices in V \setminus S in a priority queue with key a(v).
Implementing Prim’s Algorithm

More Efficient Implementation

\textbf{Prim ComputeMST}

- $E$ is the set of all edges in $G$
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for $v \notin S$, $a(v) = \min_{w \in S} c(w, v)$
for $v \notin S$, $e(v) = w$ such that $w \in S$ and $c(w, v)$ is minimum

while $S \neq V$ do
  - pick $v$ with minimum $a(v)$
  - $T = T \cup \{(e(v), v)\}$
  - $S = S \cup \{v\}$
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Maintain vertices in $V \setminus S$ in a priority queue with key $a(v)$. 
Implementing Prim’s Algorithm
More Efficient Implementation

Prim_ComputeMST

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while S \neq V do
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    T = T \cup \{(e(v), v)\}
    S = S \cup \{v\}

update arrays a and e

return the set T

Maintain vertices in V \ S in a priority queue with key a(v).
Priority Queues

Data structure to store a set $S$ of $n$ elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations

1. **makeQ**: create an empty queue
2. **findMin**: find the minimum key in $S$
3. **extractMin**: Remove $v \in S$ with smallest key and return it
4. **add($v$, $k(v)$)**: Add new element $v$ with key $k(v)$ to $S$
5. **Delete($v$)**: Remove element $v$ from $S$
6. **decreaseKey ($v$, $k'(v)$)**: decrease key of $v$ from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$
7. **meld**: merge two separate priority queues into one
Prim’s using priority queues

$E$ is the set of all edges in $G$
$S = \{1\}$
$T$ is empty (* $T$ will store edges of a MST *)
for $v \notin S$, $a(v) = \min_{w \in S} c(w, v)$
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while $S \neq V$ do
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    $S = S \cup \{v\}$
    update arrays $a$ and $e$
return the set $T$

Maintain vertices in $V \setminus S$ in a priority queue with key $a(v)$
1. Requires $O(n)$ extractMin operations
2. Requires $O(m)$ decreaseKey operations
Prim’s using priority queues

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Maintain vertices in \( V \setminus S \) in a priority queue with key \( a(v) \)

1. Requires \( \mathcal{O}(n) \) extractMin operations
2. Requires \( \mathcal{O}(m) \) decreaseKey operations
Running time of Prim’s Algorithm

\(O(n)\) \texttt{extractMin} operations and \(O(m)\) \texttt{decreaseKey} operations

1. Using standard Heaps, \texttt{extractMin} and \texttt{decreaseKey} take \(O(\log n)\) time. Total: \(O((m + n) \log n)\)

2. Using Fibonacci Heaps, \(O(\log n)\) for \texttt{extractMin} and \(O(1)\) (amortized) for \texttt{decreaseKey}. Total: \(O(n \log n + m)\).

Prim’s algorithm and Dijkstra’s algorithms are similar. Where is the difference?
Running time of Prim’s Algorithm

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Prim’s algorithm and Dijkstra’s algorithms are similar. Where is the difference?
Kruskal’s Algorithm

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Initially $E$ is the set of all edges in $G$
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while $E$ is not empty do
  choose $e \in E$ of minimum cost
  if $(T \cup \{e\}$ does not have cycles)
    add $e$ to $T$
return the set $T$

1. Presort edges based on cost. Choosing minimum can be done in $O(1)$ time
2. Do BFS/DFS on $T \cup \{e\}$. Takes $O(n)$ time
3. Total time $O(m \log m) + O(mn) = O(mn)$
Kruskal’s Algorithm

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Kruskal’s Algorithm

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    choose e ∈ E of minimum cost
    if (T ∪ {e} does not have cycles)
        add e to T
return the set T
```

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Implementing Kruskal’s Algorithm Efficiently

Kruskal\_ComputeMST

Sort edges in $E$ based on cost

$T$ is empty (* $T$ will store edges of a MST *)

each vertex $u$ is placed in a set by itself

while $E$ is not empty do

pick $e = (u, v) \in E$ of minimum cost

if $u$ and $v$ belong to different sets

add $e$ to $T$

merge the sets containing $u$ and $v$

return the set $T$

Need a data structure to check if two elements belong to same set and to merge two sets.
Implementing Kruskal’s Algorithm Efficiently

Kruskal ComputeMST

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Need a data structure to check if two elements belong to same set and to merge two sets.
MST for really sparse graphs?

Given a graph $G$ with $n$ vertices, and $n + 20$ edges, its MST can be computed in

(A) $O(n^2)$.
(B) $O(n \log n)$.
(C) $O(n \log \log n)$.
(D) $O(n \log^* n)$.
(E) $O(n)$.
Union-Find Data Structure

Data Structure

Store disjoint sets of elements that supports the following operations

1. \( \text{makeUnionFind}(S) \) returns a data structure where each element of \( S \) is in a separate set.

2. \( \text{find}(u) \) returns the name of set containing element \( u \). Thus, \( u \) and \( v \) belong to the same set if and only if \( \text{find}(u) = \text{find}(v) \).

3. \( \text{union}(A, B) \) merges two sets \( A \) and \( B \). Here \( A \) and \( B \) are the names of the sets. Typically the name of a set is some element in the set.
data structure

store disjoint sets of elements that supports the following operations

1. \texttt{makeUnionFind}(S) returns a data structure where each element of S is in a separate set

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Data Structure

Store disjoint sets of elements that supports the following operations:

1. **makeUnionFind(S)** returns a data structure where each element of S is in a separate set.

2. **find(u)** returns the *name* of set containing element u. Thus, u and v belong to the same set if and only if \( \text{find}(u) = \text{find}(v) \).

3. **union(A, B)** merges two sets A and B. Here A and B are the names of the sets. Typically the name of a set is some element in the set.
Implementing Union-Find using Arrays and Lists

Using lists

1. Each set stored as list with a name associated with the list.
2. For each element $u \in S$ a pointer to the its set. Array for pointers: $\text{component}[u]$ is pointer for $u$.
3. $\text{makeUnionFind}(S)$ takes $O(n)$ time and space.
Example
Implementing Union-Find using Arrays and Lists

1. \textbf{find}(u) reads the entry component[u]: \(O(1)\) time
2. \textbf{union}(A,B) involves updating the entries component[u] for all elements \(u\) in \(A\) and \(B\): \(O(|A| + |B|)\) which is \(O(n)\)
Implementing Union-Find using Arrays and Lists

1. \textbf{find}(u) reads the entry component[u]: \textbf{O}(1) time

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---

### Diagram

```
  s  t
    ↓    ↓
  u  w  y
    ^    ^
 v  x
    ↓    ↓
 w
    ↓    ↓
 x
    ↓    ↓
 y
    ↓    ↓
 z

 s  t
    ↓    ↓
  u  w  y
    ^    ^
 v  x
    ↓    ↓
 w
    ↓    ↓
 x
    ↓    ↓
 y
    ↓    ↓
 z
```
New Implementation

As before use component\([u]\) to store set of \(u\).

Change to union\((A, B)\):

1. with each set, keep track of its size
2. assume \(|A| \leq |B|\) for now
3. Merge the list of \(A\) into that of \(B\): \(O(1)\) time (linked lists)
4. Update component\([u]\) only for elements in the smaller set \(A\)
5. Total \(O(|A|)\) time. Worst case is still \(O(n)\).

find still takes \(O(1)\) time
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Improving the List Implementation for Union

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5. Total \(O(|A|)\) time. Worst case is still \(O(n)\).

\texttt{find} still takes \(O(1)\) time
Example

The smaller set (list) is appended to the largest set (list)
Mergers

Consider an element $x$. Assume $x$ is in a set $X$, and let $Y$ be a bigger set. After $\text{union}(X, Y)$ the size of the set containing $x$ is at least:

- **(A)** At least double what it was.
- **(B)** Same.
- **(C)** Maybe bigger, maybe the same size.
- **(D)** $|X| \times |Y|$.
- **(E)** $|X|(|Y| - |X|)$. 

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CS473  46
Fall 2014  46 / 65
Mergers

Consider starting with $n$ singletons. Consider an element $x$. The element $x$ can be participate in at most

(A) $\Theta(1)$.
(B) $\Theta(\log n)$.
(C) $\Theta(\sqrt{n})$.
(D) $\Theta(n)$.
(E) I was sworn to secrecy on this topic and as such can not answer this question

mergers where it belongs to the smaller set, throughout the execution of Union-Find.
Improving the List Implementation for Union

**Question**

Is the improved implementation provably better or is it simply a nice heuristic?

**Theorem**

Any sequence of \( k \) union operations, starting from \( \text{makeUnionFind}(S) \) on set \( S \) of size \( n \), takes at most \( O(k \log k) \).

**Corollary**

Kruskal’s algorithm can be implemented in \( O(m \log m) \) time.

Sorting takes \( O(m \log m) \) time, \( O(m) \) finds take \( O(m) \) time and \( O(n) \) unions take \( O(n \log n) \) time.
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Amortized Analysis

Why does theorem work?

**Key Observation**

\( \text{union}(A, B) \) takes \( O(|A|) \) time where \( |A| \leq |B| \). Size of new set is \( \geq 2|A| \). Cannot double too many times.
Proof of Theorem

Proof.

1. Any union operation involves at most 2 of the original one-element sets; thus at least $n - 2k$ elements have never been involved in a union.

2. Also, maximum size of any set (after $k$ unions) is $2k$.

3. $\text{union}(A, B)$ takes $O(|A|)$ time where $|A| \leq |B|$.

4. Charge each element in $A$ constant time to pay for $O(|A|)$ time.

5. How much does any element get charged?

6. If component $[v]$ is updated, set containing $v$ doubles in size.

7. component $[v]$ is updated at most $\log 2k$ times.

8. Total number of updates is $2k \log 2k = O(k \log k)$.
Better data structure

Maintain elements in a forest of *in-trees*; all elements in one tree belong to a set with root’s name.

1. **find**(u): Traverse from u to the root

Improving Worst Case Time

Better data structure

Maintain elements in a forest of *in-trees*; all elements in one tree belong to a set with root’s name.

1. **find**(*u*): Traverse from *u* to the root

2. **union**(*A*, *B*): Make root of *A* (smaller set) point to root of *B*. Takes $O(1)$ time.
Better data structure

Maintain elements in a forest of *in-trees*; all elements in one tree belong to a set with root’s name.

1. **find**(u): Traverse from *u* to the root
Details of Implementation

Each element $u \in S$ has a pointer $\text{parent}(u)$ to its ancestor.

```
makeUnionFind(S)
    for each u in S do
        parent(u) = u

find(u)
    while (parent(u) ≠ u) do
        u = parent(u)
    return u

union(component(u), component(v))
    (* parent(u) = u & parent(v) = v *)
    if (|component(u)| ≤ |component(v)|) then
        parent(u) = v
    else
        parent(v) = u
    set new component size to |component(u)| + |component(v)|
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Details of Implementation

Each element $u \in S$ has a pointer $\text{parent}(u)$ to its ancestor.

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Analysis

Theorem

The forest based implementation for a set of size $n$, has the following complexity for the various operations: \texttt{makeUnionFind} takes $O(n)$, \texttt{union} takes $O(1)$, and \texttt{find} takes $O(\log n)$.

Proof.

1. \texttt{find(u)} depends on the height of tree containing $u$.
2. Height of $u$ increases by at most 1 only when the set containing $u$ changes its name.
3. If height of $u$ increases then size of the set containing $u$ (at least) doubles.
4. Maximum set size is $n$; so height of any tree is at most $O(\log n)$.  


Further Improvements: Path Compression

Observation

Consecutive calls of \texttt{find(u)} take $O(\log n)$ time each, but they traverse the same sequence of pointers.

Idea: Path Compression

Make all nodes encountered in the \texttt{find(u)} point to root.
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Make all nodes encountered in the \( \text{find}(u) \) point to root.
Path Compression: Example
Path Compression

```plaintext

\( \text{find}(u) : \)
  \( \text{if } (\text{parent}(u) \neq u) \text{ then} \)
  \( \text{parent}(u) = \text{find}(\text{parent}(u)) \)
  \( \text{return } \text{parent}(u) \)

Question

Does Path Compression help?

Yes!

Theorem

With Path Compression, \( k \) operations (\text{find} and/or \text{union}) take \( O(k\alpha(k, \min\{k, n\})) \) time where \( \alpha \) is the inverse Ackermann function.
Path Compression

\[ \text{find}(u) : \]

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\text{if } (\text{parent}(u) \neq u) & \text{ then} \\
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Ackermann and Inverse Ackermann Functions

Ackermann function $A(m, n)$ defined for $m, n \geq 0$ recursively

$$A(m, n) = \begin{cases} 
  n + 1 & \text{if } m = 0 \\
  A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\
  A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0
\end{cases}$$

$A(3, n) = 2^{n+3} - 3$
$A(4, 3) = 2^{65536} - 3$

$\alpha(m, n)$ is inverse Ackermann function defined as

$$\alpha(m, n) = \min\{i \mid A(i, \lfloor m/n \rfloor) \geq \log_2 n\}$$

For all practical purposes $\alpha(m, n) \leq 5$
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Amazing result:

**Theorem (Tarjan)**

For **Union-Find**, any data structure in the pointer model requires $\Omega(m^{\alpha(m, n)})$ time for $m$ operations.
Running time of Kruskal’s Algorithm

Using Union-Find data structure:

1. \( O(m) \) find operations (two for each edge)
2. \( O(n) \) union operations (one for each edge added to \( T \))
3. Total time: \( O(m \log m) \) for sorting plus \( O(m\alpha(n)) \) for union-find operations. Thus \( O(m \log m) \) time despite the improved Union-Find data structure.
Prim’s algorithm using Fibonacci heaps: $O(n \log n + m)$.
If $m$ is $O(n)$ then running time is $\Omega(n \log n)$.

Question
Is there a linear time ($O(m + n)$ time) algorithm for MST?

1. $O(m \log^* m)$ time Fredman and Tarjan [1987].
2. $O(m + n)$ time using bit operations in RAM model Fredman and Willard [1994].
3. $O(m + n)$ expected time (randomized algorithm) Karger et al. [1995].
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