Part I

Exponentiation, Binary Search
Exponentiation

Input Two numbers: $a$ and integer $n \geq 0$

Goal Compute $a^n$

Obvious algorithm:

\begin{verbatim}
SlowPow(a,n):
    x = 1;
    for i = 1 to n do
        x = x*a
    Output x
\end{verbatim}

$O(n)$ multiplications.
Exponentiation

Input  Two numbers:  \( a \) and integer  \( n \geq 0 \)
Goal   Compute  \( a^n \)

Obvious algorithm:

**SlowPow** \((a,n)\):

\[
x = 1; \\
\text{for } i = 1 \text{ to } n \text{ do} \\
x = x*a \\
\text{Output } x
\]

\(O(n)\)  multiplications.
How many bits...

Let $a > 1$ and $n > 1$ be two integer numbers. Representing $a^n$ in base 2 requires

(A) $O(\log a + \log n)$ bits.
(B) $O(n \log a)$ bits.
(C) $O(a \log n)$ bits.
(D) $O(\log a \log n)$ bits.
(E) $O\left((\log a)^{\log n}\right)$ bits.
Fast Exponentiation

Observation: \(a^n = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} - \lfloor n/2 \rfloor}\).

\[
\text{FastPow}(a, n): \\
\text{if (n = 0) return 1} \\
x = \text{FastPow}(a, \lfloor n/2 \rfloor) \\
x = x \times x \\
\text{if (n is odd) then} \\
x = x \times a \\
\text{return x}
\]

\(T(n)\): number of multiplications for \(n\)

\[T(n) \leq T(\lfloor n/2 \rfloor) + 2\]

\(T(n) = \Theta(\log n)\)
Fast Exponentiation

Observation: \( a^n = a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil} = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} - \lfloor n/2 \rfloor} \).

**FastPow**(a, n):

\[
\text{if } (n = 0) \text{ return } 1 \\
x = \text{FastPow}(a, \lfloor n/2 \rfloor) \\
x = x \times x \\
\text{if } (n \text{ is odd}) \text{ then} \\
x = x \times a \\
\text{return } x
\]

\( T(n) \): number of multiplications for \( n \)

\[
T(n) \leq T(\lfloor n/2 \rfloor) + 2
\]

\( T(n) = \Theta(\log n) \)
Fast Exponentiation

Observation:  \( a^n = a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil} = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} - \lfloor n/2 \rfloor} \).

\( \text{FastPow}(a, n) \):

1. if \( (n = 0) \) return 1
2. \( x = \text{FastPow}(a, \lfloor n/2 \rfloor) \)
3. \( x = x \times x \)
4. if \( (n \text{ is odd}) \) then
   1. \( x = x \times a \)
5. return \( x \)

\( T(n) \): number of multiplications for \( n \)

\[ T(n) \leq T(\lfloor n/2 \rfloor) + 2 \]

\( T(n) = \Theta(\log n) \)
Fast Exponentiation

Observation: \( a^n = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} - \lfloor n/2 \rfloor} \).

**FastPow** \((a, n)\):

- if \((n = 0)\) return 1
- \(x = \text{FastPow}(a, \lfloor n/2 \rfloor)\)
- \(x = x \times x\)
- if \((n \text{ is odd})\) then
  - \(x = x \times a\)
- return \(x\)

\(T(n)\): number of multiplications for \(n\)

\[ T(n) \leq T(\lfloor n/2 \rfloor) + 2 \]

\(T(n) = \Theta(\log n)\)
Fast Exponentiation

Observation: \(a^n = a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil} = a^{\lfloor n/2 \rfloor} a^{\lfloor n/2 \rfloor} a^{\lceil n/2 \rceil - \lfloor n/2 \rfloor}\).

\(\text{FastPow}(a, n)\):
\[
\text{if } (n = 0) \text{ return } 1 \\
x = \text{FastPow}(a, \lfloor n/2 \rfloor) \\
x = x \times x \\
\text{if } (n \text{ is odd}) \text{ then} \\
x = x \times a \\
\text{return } x
\]

\(T(n)\): number of multiplications for \(n\)

\[T(n) \leq T(\lfloor n/2 \rfloor) + 2\]

\(T(n) = \Theta(\log n)\)
Question: Is SlowPow() a polynomial time algorithm? FastPow?
Input size: $O(\log a + \log n)$
Output size: $O(n \log a)$.

Not necessarily polynomial in input size!

Both SlowPow and FastPow are polynomial in output size.
Question: Is \texttt{SlowPow()} a polynomial time algorithm? \texttt{FastPow}?

Input size: \(O(\log a + \log n)\)

Output size: \(O(n \log a)\).

Not necessarily polynomial in input size!

Both \texttt{SlowPow} and \texttt{FastPow} are polynomial in output size.
Question: Is \texttt{SlowPow()} a polynomial time algorithm? \texttt{FastPow}?

Input size: \(O(\log a + \log n)\)

Output size: \(O(n \log a)\).

Not necessarily polynomial in input size!

Both \texttt{SlowPow} and \texttt{FastPow} are polynomial in output size.
Complexity of Exponentiation

**Question:** Is \texttt{SlowPow()} a polynomial time algorithm? \texttt{FastPow}?

- Input size: \(O(\log a + \log n)\)
- Output size: \(O(n \log a)\).

Not necessarily polynomial in input size!

Both \texttt{SlowPow} and \texttt{FastPow} are polynomial in output size.
26 \ mod \ 7 \ is? \\
(A) 0 \\
(B) 1 \\
(C) 3 \\
(D) 5 \\
(E) 7
Exponentiation modulo a given number

Exponentiation in applications:

**Input**  Three integers:  \( a, n \geq 0, p \geq 2 \) (typically a prime)

**Goal**  Compute \( a^n \mod p \)

**Input size:**  \( \Theta(\log a + \log n + \log p) \)
**Output size:**  \( O(\log p) \) and hence polynomial in input size.

**Observation:** \( xy \mod p = ((x \mod p)(y \mod p)) \mod p \)
Exponentiation modulo a given number

Exponentiation in applications:

Input  Three integers: $a, n \geq 0, p \geq 2$ (typically a prime)
Goal  Compute $a^n \mod p$

Input size: $\Theta(\log a + \log n + \log p)$
Output size: $O(\log p)$ and hence polynomial in input size.

Observation: $xy \mod p = ((x \mod p)(y \mod p)) \mod p$
Exponentiation modulo a given number

Exponentiation in applications:

Input Three integers: \( a, n \geq 0, p \geq 2 \) (typically a prime)

Goal Compute \( a^n \mod p \)

Input size: \( \Theta(\log a + \log n + \log p) \)
Output size: \( O(\log p) \) and hence polynomial in input size.

Observation: \( xy \mod p = ((x \mod p)(y \mod p)) \mod p \)
Exponentiation modulo a given number

**Input**  Three integers: \( a, n \geq 0, p \geq 2 \) (typically a prime)

**Goal**  Compute \( a^n \mod p \)

**FastPowMod** \((a, n, p)\):

1. if \((n = 0)\) return 1
2. \(x = \text{FastPowMod}(a, \lfloor n/2 \rfloor, p)\)
3. \(x = x \times x \mod p\)
4. if \((n\text{ is odd})\)
   \(x = x \times a \mod p\)
5. return \(x\)

**FastPowMod** is a polynomial time algorithm. **SlowPowMod** is not (why?).
Exponentiation modulo a given number

Input  Three integers:  \( a, n \geq 0, p \geq 2 \) (typically a prime)

Goal  Compute \( a^n \mod p \)

\[
\text{FastPowMod}(a, n, p):
\begin{align*}
\text{if (} n = 0 \text{) return 1} \\
x &= \text{FastPowMod}(a, \lfloor n/2 \rfloor, p) \\
x &= x \times x \mod p \\
\text{if (} n \text{ is odd)} \\
x &= x \times a \mod p \\
\text{return } x
\end{align*}
\]

\textbf{FastPowMod} is a polynomial time algorithm. \textbf{SlowPowMod} is not (why?).
Binary Search in Sorted Arrays

Input  Sorted array \( A \) of \( n \) numbers and number \( x \)

Goal   Is \( x \) in \( A \)?

\[
\text{BinarySearch}(A[a..b], x):
\]
\[
\text{if } (b - a < 0) \text{ return NO}
\]
\[
\text{mid} = A[\lfloor (a + b)/2 \rfloor]
\]
\[
\text{if } (x = \text{mid}) \text{ return YES}
\]
\[
\text{if } (x < \text{mid})
\]
\[
\text{return BinarySearch}(A[a..\lfloor (a + b)/2 \rfloor - 1], x)
\]
\[
\text{else}
\]
\[
\text{return BinarySearch}(A[\lfloor (a + b)/2 \rfloor + 1..b], x)
\]

Analysis: \( T(n) = T(\lfloor n/2 \rfloor) + O(1) \). \( T(n) = O(\log n) \).

Observation: After \( k \) steps, size of array left is \( n/2^k \).
Binary Search in Sorted Arrays

Input: Sorted array $A$ of $n$ numbers and number $x$

Goal: Is $x$ in $A$?

**BinarySearch**($A[a..b], x$):

1. if ($b - a < 0$) return NO
2. $mid = A[(a + b)/2]]$
3. if ($x = mid$) return YES
4. if ($x < mid$)
   - return **BinarySearch**($A[a..[(a + b)/2] - 1], x$)
5. else
   - return **BinarySearch**($A[[(a + b)/2] + 1..b], x$)

Analysis: $T(n) = T(\lfloor n/2 \rfloor) + O(1)$. $T(n) = O(\log n)$.

Observation: After $k$ steps, size of array left is $n/2^k$
Input Sorted array $A$ of $n$ numbers and number $x$

Goal Is $x$ in $A$?

**BinarySearch**($A[a..b]$, $x$):

1. if $(b - a < 0)$ return NO
2. mid = $A[⌊(a + b)/2⌋]$
3. if $(x = \text{mid})$ return YES
4. if $(x < \text{mid})$
   - return **BinarySearch**($A[a..⌊(a + b)/2⌋ - 1]$, $x$)
5. else
   - return **BinarySearch**($A[⌊(a + b)/2⌋ + 1..b]$, $x$)

Analysis: $T(n) = T(⌊n/2⌋) + O(1)$. $T(n) = O(\log n)$.

Observation: After $k$ steps, size of array left is $n/2^k$
Another common use of binary search

1. **Optimization version:** find solution of best (say minimum) value
2. **Decision version:** is there a solution of value at most a given value $v$?

Reduce optimization to decision (may be easier to think about):

1. Given instance $I$ compute upper bound $U(I)$ on best value
2. Compute lower bound $L(I)$ on best value
3. Do binary search on interval $[L(I), U(I)]$ using decision version as black box
4. $O(\log(U(I) - L(I)))$ calls to decision version if $U(I), L(I)$ are integers
Another common use of binary search

1. **Optimization version:** find solution of best (say minimum) value
2. **Decision version:** is there a solution of value at most a given value $v$?

Reduce optimization to decision (may be easier to think about):

1. Given instance $I$ compute upper bound $U(I)$ on best value
2. Compute lower bound $L(I)$ on best value
3. Do binary search on interval $[L(I), U(I)]$ using decision version as black box
4. $O(\log(U(I) - L(I)))$ calls to decision version if $U(I), L(I)$ are integers
Example

1. **Problem**: shortest paths in a graph.

2. **Decision version**: given $G$ with non-negative integer edge lengths, nodes $s, t$ and bound $B$, is there an $s$-$t$ path in $G$ of length at most $B$?

3. **Optimization version**: find the length of a shortest path between $s$ and $t$ in $G$.

**Question**: given a black box algorithm for the decision version, can we obtain an algorithm for the optimization version?
Example continued

Question: given a black box algorithm for the decision version, can we obtain an algorithm for the optimization version?

1. Let $U$ be maximum edge length in $G$.
2. Minimum edge length is $L$.
3. $s$-$t$ shortest path length is at most $(n - 1)U$ and at least $L$.
5. $O(\log((n - 1)U - L))$ calls to the decision problem algorithm sufficient. Polynomial in input size.
Which of the following facts is wrong?

(A) Valhalla is a city in Sweden.
(B) The term “dynamic programming” was invented in the 1940s by Richard Bellman (from the Bellman-Ford fame).
(C) In 1202, Fibonacci wrote a book introducing the Arabic numeral system to Europe.
(D) Vandalia was once the capital of Illinois,
(E) Only 39% of U.S. citizens have a valid passport.
Part II

Introduction to Dynamic Programming
Recursion

Reduction:
Reduce one problem to another

Recursion
A special case of reduction
1. reduce problem to a *smaller* instance of *itself*
2. self-reduction

1. Problem instance of size $n$ is reduced to one or more instances of size $n - 1$ or less.
2. For termination, problem instances of small size are solved by some other method as base cases.
Recursion

Reduction:
Reduce one problem to another

Recursion
A special case of reduction

1. reduce problem to a \textit{smaller} instance of \textit{itself}
2. self-reduction

1. Problem instance of size $n$ is reduced to one or more instances of size $n - 1$ or less.
2. For termination, problem instances of small size are solved by some other method as base cases.
Recursion in Algorithm Design

1. **Tail Recursion**: problem reduced to a single recursive call after some work. Easy to convert algorithm into iterative or greedy algorithms. Examples: Interval scheduling, MST algorithms, etc.

2. **Divide and Conquer**: Problem reduced to multiple independent sub-problems that are solved separately. Conquer step puts together solution for bigger problem. Examples: Closest pair, deterministic median selection, quick sort.

3. **Dynamic Programming**: problem reduced to multiple (typically) dependent or overlapping sub-problems. Use memoization to avoid recomputation of common solutions leading to iterative bottom-up algorithm.
Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n - 1) + F(n - 2) \text{ and } F(0) = 0, F(1) = 1.$$  

These numbers have many interesting and amazing properties.

A journal *The Fibonacci Quarterly*!

1. $$F(n) = \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}}$$ where $\phi$ is the golden ratio 
   
   $$(1 + \sqrt{5})/2 \approx 1.618.$$  

2. $$\lim_{n \to \infty} \frac{F(n + 1)}{F(n)} = \phi$$
How many bits?

Consider the $n$th Fibonacci number $F(n)$. Writing the number $F(n)$ in base 2 requires

(A) $\Theta(n^2)$ bits.

(B) $\Theta(\log n)$ bits.

(C) $\Theta(\log \log n)$ bits.

(D) $\Theta(n)$ bits.

(E) IDK.
Question: Given \( n \), compute \( F(n) \).

\[
Fib(n) : \\
\text{if } (n = 0) \\
\quad \text{return } 0 \\
\text{else if } (n = 1) \\
\quad \text{return } 1 \\
\text{else} \\
\quad \text{return } Fib(n - 1) + Fib(n - 2)
\]

Running time? Let \( T(n) \) be the number of additions in Fib(n).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \quad \text{and} \quad T(0) = T(1) = 0
\]
Question: Given \( n \), compute \( F(n) \).

\[
Fib(n) : \\
\text{if } (n = 0) \\
\quad \text{return } 0 \\
\text{else if } (n = 1) \\
\quad \text{return } 1 \\
\text{else} \\
\quad \text{return } Fib(n - 1) + Fib(n - 2)
\]

Running time? Let \( T(n) \) be the number of additions in \( \text{Fib}(n) \).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0
\]
Question: Given \( n \), compute \( F(n) \).

\[ \text{Fib}(n): \]
\[
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{else if } (n = 1) & \quad \text{return } 1 \\
\text{else} & \quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]

Running time? Let \( T(n) \) be the number of additions in \( \text{Fib}(n) \).

\[ T(n) = T(n - 1) + T(n - 2) + 1 \text{ and } T(0) = T(1) = 0 \]
Recursive Algorithm for Fibonacci Numbers

Question: Given \( n \), compute \( F(n) \).

\[
Fib(n) :
\]
\[
\text{if } (n = 0) \quad \text{return } 0
\]
\[
\text{else if } (n = 1) \quad \text{return } 1
\]
\[
\text{else} \quad \text{return } Fib(n - 1) + Fib(n - 2)
\]

Running time? Let \( T(n) \) be the number of additions in \( Fib(n) \).

\[
T(n) = T(n - 1) + T(n - 2) + 1 \quad \text{and} \quad T(0) = T(1) = 0
\]

Roughly same as \( F(n) \)

\[
T(n) = \Theta(\phi^n)
\]

The number of additions is exponential in \( n \). Can we do better?
Running time of binom?

```
binom(t, b)  // computes \( \binom{t}{b} \)
// Using the identity: \( \binom{t}{b} = \binom{t-1}{b-1} + \binom{t-1}{b} \)
    if b = t then return 1
    if b = 0 then return 0
    return binom(t - 1, b - 1) + binom(t - 1, b).
```

Assuming each arithmetic operation takes \( O(1) \) time, the running time of `binom(n, \lfloor n/2 \rfloor)` is

(A) \( \Theta(1) \).

(B) \( \Theta(n) \).

(C) \( \Theta(n \log n) \).

(D) \( \Theta(n^2) \).

(E) \( \Theta \left( \binom{n}{\lfloor n/2 \rfloor} \right) \).
An iterative algorithm for Fibonacci numbers

\[ \text{FibIter}(n) : \]
\[
\begin{align*}
&\text{if } (n = 0) \text{ then} \\
&\quad \text{return } 0 \\
&\text{if } (n = 1) \text{ then} \\
&\quad \text{return } 1 \\
&F[0] = 0 \\
&F[1] = 1 \\
&\text{for } i = 2 \text{ to } n \text{ do} \\
&\quad F[i] \leftarrow F[i - 1] + F[i - 2] \\
&\text{return } F[n]
\end{align*}
\]

What is the running time of the algorithm? \( O(n) \) additions.
An iterative algorithm for Fibonacci numbers

\[
\text{FibIter}(n): \\
\begin{align*}
&\text{if } (n = 0) \text{ then} \\
&\quad \text{return } 0 \\
&\text{if } (n = 1) \text{ then} \\
&\quad \text{return } 1 \\
&F[0] = 0 \\
&F[1] = 1 \\
&\text{for } i = 2 \text{ to } n \text{ do} \\
&\quad F[i] &\leftarrow F[i - 1] + F[i - 2] \\
&\text{return } F[n]
\end{align*}
\]

What is the running time of the algorithm? \(O(n)\) additions.
An iterative algorithm for Fibonacci numbers

\textbf{FibIter}(n):
\begin{align*}
&\text{if } (n = 0) \text{ then} \\
&\quad \text{return } 0 \\
&\text{if } (n = 1) \text{ then} \\
&\quad \text{return } 1 \\
&F[0] = 0 \\
&F[1] = 1 \\
&\text{for } i = 2 \text{ to } n \text{ do} \\
&\quad F[i] \leftarrow F[i - 1] + F[i - 2] \\
&\text{return } F[n]
\end{align*}

What is the running time of the algorithm? $O(n)$ additions.
What is the difference?

1. Recursive algorithm is computing the same numbers again and again.

2. Iterative algorithm is storing computed values and building bottom up the final value. **Memoization.**

**Dynamic Programming:**

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
What is the difference?

1. Recursive algorithm is computing the same numbers again and again.
2. Iterative algorithm is storing computed values and building bottom up the final value. Memoization.

Dynamic Programming:
Finding a recursion that can be effectively/efficiently memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
What is the difference?

1. Recursive algorithm is computing the same numbers again and again.
2. Iterative algorithm is storing computed values and building bottom up the final value. **Memoization.**

**Dynamic Programming:**

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of sub-problems is polynomial in input size.
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) :=
\begin{array}{l}
\text{if } (n = 0) \\
\quad \text{return } 0 \\
\text{if } (n = 1) \\
\quad \text{return } 1 \\
\text{if } (\text{Fib}(n) \text{ was previously computed}) \\
\quad \text{return stored value of Fib}(n) \\
\text{else} \\
\quad \text{return Fib}(n - 1) + \text{Fib}(n - 2)
\end{array}
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\texttt{Fib}(n):

\begin{verbatim}
if (n = 0)
    return 0
if (n = 1)
    return 1
if (Fib(n) was previously computed)
    return stored value of Fib(n)
else
    return Fib(n − 1) + Fib(n − 2)
\end{verbatim}

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) :
\]

\[
\begin{align*}
\text{if } (n = 0) & \quad \text{return 0} \\
\text{if } (n = 1) & \quad \text{return 1} \\
\text{if } (\text{Fib}(n) \text{ was previously computed}) & \quad \text{return stored value of Fib}(n) \\
\text{else} & \quad \text{return Fib}(n - 1) + \text{Fib}(n - 2)
\end{align*}
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic Memoization

Can we convert recursive algorithm into an efficient algorithm without explicitly doing an iterative algorithm?

\[
\text{Fib}(n) : \\
\quad \text{if } (n = 0) \\
\quad \quad \text{return } 0 \\
\quad \text{if } (n = 1) \\
\quad \quad \text{return } 1 \\
\quad \text{if } (\text{Fib}(n) \text{ was previously computed}) \\
\quad \quad \text{return } \text{stored value of Fib}(n) \\
\quad \text{else} \\
\quad \quad \text{return } \text{Fib}(n - 1) + \text{Fib}(n - 2)
\]

How do we keep track of previously computed values?
Two methods: explicitly and implicitly (via data structure)
Automatic explicit memoization

Initialize table/array $M$ of size $n$ such that $M[i] = -1$ for $i = 0, \ldots, n$.

Fib$(n)$:

if ($n = 0$)
    return 0
if ($n = 1$)
    return 1
if ($M[n] \neq -1$) (* $M[n]$ has stored value of Fib$(n)$ *)
    return $M[n]$
$M[n] \leftarrow$ Fib$(n - 1) +$ Fib$(n - 2)$
return $M[n]$

Need to know upfront the number of subproblems to allocate memory
Automatic explicit memoization

Initialize table/array $M$ of size $n$ such that $M[i] = -1$ for $i = 0, \ldots, n$.

$Fib(n)$:

if $(n = 0)$
    return 0
if $(n = 1)$
    return 1
if $(M[n] \neq -1)$ (* $M[n]$ has stored value of $Fib(n)$ *)
    return $M[n]$

$M[n] \leftarrow Fib(n - 1) + Fib(n - 2)$
return $M[n]$

Need to know upfront the number of subproblems to allocate memory
Automatic implicit memoization

Initialize a (dynamic) dictionary data structure \( D \) to empty

\[ \text{Fib}(n): \]

\[
\begin{align*}
\text{if } (n = 0) & \quad \text{return } 0 \\
\text{if } (n = 1) & \quad \text{return } 1 \\
\text{if } (n \text{ is already in } D) & \quad \text{return \ value \ stored \ with \ } n \text{ in } D \\
\text{val} & \leftarrow \text{Fib}(n - 1) + \text{Fib}(n - 2) \\
\text{Store } (n, \text{val}) \text{ in } D \\
\text{return } \text{val}
\end{align*}
\]
Explicit vs Implicit Memoization

1. Explicit memoization or iterative algorithm preferred if one can analyze problem ahead of time. Allows for efficient memory allocation and access.

2. Implicit and automatic memoization used when problem structure or algorithm is either not well understood or in fact unknown to the underlying system.
   - Need to pay overhead of data-structure.
   - Functional languages such as LISP automatically do memoization, usually via hashing based dictionaries.
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

1. input is $n$ and hence input size is $\Theta(\log n)$
2. output is $F(n)$ and output size is $\Theta(n)$. Why?
3. Hence output size is exponential in input size so no polynomial time algorithm possible!
4. Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?
5. Running time of recursive algorithm is $O(n\phi^n)$ but can in fact shown to be $O(\phi^n)$ by being careful. Doubly exponential in input size and exponential even in output size.
Is the iterative algorithm a *polynomial* time algorithm? Does it take $O(n)$ time?

1. input is $n$ and hence input size is $\Theta(\log n)$

2. output is $F(n)$ and output size is $\Theta(n)$. Why?

3. Hence output size is exponential in input size so no polynomial time algorithm possible!

4. Running time of iterative algorithm: $\Theta(n)$ additions but number sizes are $O(n)$ bits long! Hence total time is $O(n^2)$, in fact $\Theta(n^2)$. Why?

5. Running time of recursive algorithm is $O(n\phi^n)$ but can in fact shown to be $O(\phi^n)$ by being careful. Doubly exponential in input size and exponential even in output size.
How many distinct calls does \( \text{binom}(n, \lfloor n/2 \rfloor) \) makes during its recursive execution?

(A) \( \Theta(1) \).

(B) \( \Theta(n) \).

(C) \( \Theta(n \log n) \).

(D) \( \Theta(n^2) \).

(E) \( \Theta \left( \binom{n}{\lfloor n/2 \rfloor} \right) \).

That is, if the algorithm calls recursively \( \text{binom}(17, 5) \) about 5000 times during the computation, we count this is a single distinct call.
Running time of memoized binom?

\[
\begin{align*}
D: & \text{ Initially an empty dictionary.} \\
\text{binomM}(t, b) & \quad \text{// computes } \binom{t}{b} \\
\text{if } b = t & \text{ then return } 1 \\
\text{if } b = 0 & \text{ then return } 0 \\
\text{if } D[t, b] \text{ is defined} & \text{ then return } D[t, b] \\
D[t, b] & \leftarrow \text{binomM}(t - 1, b - 1) + \text{binomM}(t - 1, b) \\
\text{return } D[t, b]
\end{align*}
\]

Assuming that every arithmetic operation takes \( O(1) \) time, What is the running time of \text{binomM}(n, \lfloor n/2 \rfloor)\? 

\( (A) \ \Theta(1). \)

\( (B) \ \Theta(n). \)

\( (C) \ \Theta(n^2). \)

\( (D) \ \Theta(n^3). \)

\( (E) \ \Theta\left(\binom{n}{\lfloor n/2 \rfloor}\right). \)
**Maximum Independent Set in a Graph**

**Definition**

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in $S$. That is, if $u, v \in S$ then $(u, v) \not\in E$.

Some independent sets in graph above:
Maximum Independent Set Problem

**Input**  Graph $G = (V, E)$

**Goal**  Find maximum sized independent set in $G$
Maximum Weight Independent Set Problem

Input  Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal  Find maximum weight independent set in $G$
No one knows an efficient (polynomial time) algorithm for this problem.

Problem is **NP-Complete** and it is *believed* that there is no polynomial time algorithm.

**Brute-force algorithm:**

Try all subsets of vertices.
Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

\[ \text{MaxIndSet}(G = (V, E)) : \]
\[
\text{max} = 0 \\
\text{for each subset } S \subseteq V \text{ do} \\
\quad \text{check if } S \text{ is an independent set} \\
\quad \text{if } S \text{ is an independent set and } w(S) > \text{max} \text{ then} \\
\quad \quad \text{max} = w(S) \\
\text{Output } \text{max}
\]

Running time: suppose \( G \) has \( n \) vertices and \( m \) edges

1. \( 2^n \) subsets of \( V \)
2. checking each subset \( S \) takes \( O(m) \) time
3. total time is \( O(m2^n) \)
Brute-force enumeration

Algorithm to find the size of the maximum weight independent set.

\[
\text{MaxIndSet}(G = (V, E)):
\]
\[
\text{max} = 0
\]
\[
\text{for each subset } S \subseteq V \text{ do}
\]
\[
\text{check if } S \text{ is an independent set}
\]
\[
\text{if } S \text{ is an independent set and } w(S) > \text{max} \text{ then}
\]
\[
\text{max} = w(S)
\]
\[
\text{Output } \text{max}
\]

Running time: suppose \( G \) has \( n \) vertices and \( m \) edges

1. \( 2^n \) subsets of \( V \)
2. checking each subset \( S \) takes \( O(m) \) time
3. total time is \( O(m2^n) \)
A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$. For a vertex $u$ let $N(u)$ be its neighbors.

**Observation**

$v_n$: Vertex in the graph.

One of the following two cases is true

- **Case 1** $v_n$ is in some maximum independent set.
- **Case 2** $v_n$ is in no maximum independent set.

**RecursiveMIS**($G$):

1. if $G$ is empty then Output 0
2. $a = \text{RecursiveMIS}(G - v_n)$
3. $b = w(v_n) + \text{RecursiveMIS}(G - v_n - N(v_n))$
4. Output $\max(a, b)$
A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$.
For a vertex $u$ let $N(u)$ be its neighbors.

Observation

$v_n$: Vertex in the graph.
One of the following two cases is true

Case 1 $v_n$ is in some maximum independent set.
Case 2 $v_n$ is in no maximum independent set.

RecursiveMIS($G$):
if $G$ is empty then Output 0
$a = \text{RecursiveMIS}(G - v_n)$
$b = w(v_n) + \text{RecursiveMIS}(G - v_n - N(v_n))$
Output $\max(a, b)$
A Recursive Algorithm

Let $V = \{v_1, v_2, \ldots, v_n\}$.
For a vertex $u$ let $N(u)$ be its neighbors.

**Observation**

$v_n$: Vertex in the graph.

One of the following two cases is true

**Case 1** $v_n$ is in some maximum independent set.

**Case 2** $v_n$ is in no maximum independent set.

**RecursiveMIS**($G$):

*if G is empty then Output 0

$a = \text{RecursiveMIS}(G - v_n)$

$b = w(v_n) + \text{RecursiveMIS}(G - v_n - N(v_n))$

Output $\max(a, b)$
Running time:

\[ T(n) = T(n - 1) + T\left(n - 1 - \text{deg}(v_n)\right) + O(1 + \text{deg}(v_n)) \]

where \( \text{deg}(v_n) \) is the degree of \( v_n \). \( T(0) = T(1) = 1 \) is base case.

Worst case is when \( \text{deg}(v_n) = 0 \) when the recurrence becomes

\[ T(n) = 2T(n - 1) + O(1) \]

Solution to this is \( T(n) = O(2^n) \).
Recursive algorithm generates a tree of computation where each node is a smaller problem (subproblem).

Simple recursive algorithm computes/explores the whole tree blindly in some order.

Backtrack search is a way to explore the tree intelligently to prune the search space.

1. Some subproblems may be so simple that we can stop the recursive algorithm and solve it directly by some other method.
2. Memoization to avoid recomputing same problem.
3. Stop the recursion at a subproblem if it is clear that there is no need to explore further.
4. Leads to a number of heuristics that are widely used in practice although the worst case running time may still be exponential.